Strategic Investment Evaluation*

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Abstract

We study the interaction of incentives to free-ride on information acquisition and strategically delay irreversible investment in environments in which multiple firms evaluate an investment opportunity. In our model, two firms decide how quickly to privately obtain information about the profitability of a project, and when (if ever) to publicly invest in it. Multiple equilibria exist, differing with respect to how much information firms acquire as well as how quickly they invest. The equilibrium which maximizes aggregate payoffs features asymmetric play with distinct leader and follower roles when firms are patient, but features symmetric play when firms are impatient and information acquisition costs are sufficiently high.

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1 Introduction

In many economic settings, decision makers may strategically delay irreversible action in order to learn from the actions of others. For instance, oil firms can delay drilling on

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leased tracts to learn from the drilling decisions of firms on nearby tracts,[1] and venture capitalists can delay investing in startups to learn from the funding decisions of other investors.[2] More generally, incentives for strategic delay arise whenever information about payoffs is dispersed, opportunities are non-rival, and decision-makers may freely time their actions.

Our starting point is the observation that in many applications, a decision maker's private information is the result of costly information-acquisition activities. For instance, oil firms conduct seismic surveys to estimate the extent of oil deposits on a tract, and venture capitalists perform due diligence to gauge the quality of a startup's product and management team. Strategic incentives then shape both how much information is *produced* through private effort, as well as how much is *aggregated* through public actions.

A key insight from the literature on experimentation in teams is that when information acquisition is costly, decision makers tend to free-ride, or inefficiently reduce their rate of information acquisition. We build on that insight by assuming, in a departure from existing work, that learning is both private and imperfect, and that decision-makers reveal what they know only by irreversible action. These features create an incentive for players to strategically delay acting on good news in addition to, or instead of, free-riding on the acquisition of news. Our model provides a tractable framework for studying the equilibrium interplay of incentives for free-riding and strategic delay.

In our model, two firms have the opportunity to invest in a nonrival risky project. Firms may dynamically exert variable costly effort, a process we call "prospecting", for the chance of receiving a binary signal which is informative about the project's value. Each firm can acquire at most one signal, and signals are conditionally i.i.d. As a result, aggregating signals from multiple firms yields information about the profitability of investment beyond what any one firm could learn. Any information a firm acquires through prospecting is private but investment is public.

We show that there are exactly three perfect Bayesian equilibria of our model. In

[1]See Hendricks and Kovenock (1989) for a discussion of incentives for social learning in offshore oil drilling and Hendricks and Porter (1996) for empirical evidence of strategic delay in this setting.

[2]Paul Graham, a prominent entrepreneur and venture capitalist, has discussed the importance of social learning among venture capitalists: "The biggest component in most investors' opinion of you is the opinion of other investors... When one investor wants to invest in you, that makes other investors want to, which makes others want to, and so on"(Graham 2013).

the unique symmetric equilibrium, each firm prospects as intensively as possible until a cutoff time, after which it abandons prospecting forever if it has not seen investment by the other firm. If at any time before the cutoff a firm receives a positive signal, it invests without delay. This equilibrium exhibits no free-riding or investment delay.

There are also two asymmetric "leader-follower" equilibria. In these equilibria, one firm takes the role of a leader, prospecting until acquiring a signal and then investing without delay if the signal is positive. Meanwhile the remaining firm follows the leader by either free-riding on the leader's prospecting efforts, delaying investment after acquiring a signal, or both. The mix of the two behaviors depends on the cost of prospecting: investment delay arises when costs are low, free-riding emerges when costs are high, and for intermediate costs delay is followed by eventual free-riding. In the low-cost regime, not only is there no free-riding, but the follower spends more time prospecting than it would have in the symmetric equilibrium.

In contrast to existing models of free-riding and investment delay, neither equilibrium generates unambiguously larger amounts of social learning. In general, the symmetric equilibrium produces more information early on, while the leader-follower equilibrium produces more at later times. As a result, either equilibrium can generate higher total payoffs, depending on model parameters. We show that when firms are patient, the leader-follower equilibrium generates higher aggregate payoffs, while when firms are impatient and prospecting costs are sufficiently high, the symmetric equilibrium is superior.

The remainder of the paper is organized as follows. Section 1.1 surveys related literature. Section 2 describes the model. Section 3 characterizes the set of perfect Bayesian equilibria of the model. Section 4 compares payoffs across equilibria. Section 5 concludes.

1.1 Related literature

Our paper is most closely connected to models of collective experimentation, in particular Bonatti and Hörner (2011, 2017), Bolton and Harris (1999), Keller, Rady, and Cripps (2005), Keller and Rady (2010, 2015), and Dong (2018).[3] These papers study environments in which effort simultaneously dictates both the production and aggregation of information. This linkage is a key feature of the canonical bandit

[3] See Hörner and Skrzypacz (2017) for an excellent survey of this literature.

experimentation framework, in which players learn by monitoring the returns to incrementally investing effort in a project. Our paper departs from this literature by separating learning from the payoffs generated by a project. This separation allows us to make learning completely private, with information aggregation instead associated with a distinct decision to collect payoffs.

Our model builds most directly on on the work of Bonatti and Hörner (2011) (hereafter BH), who study strategic experimentation with private effort and learning, in a setting where signals arrive via a Poisson good news process with perfectly informative breakthroughs.[4] A key dynamic in both their model and ours is a gradual deterioration of each player's beliefs due to continued inaction by another player, which is taken as a negative signal about their private information. Methodologically, our model differs by modeling negative signals as arriving discretely rather than continuously, simplifying equilibrium characterizations by avoiding belief divergences following deviations from equilibrium effort. And conceptually, it differs by assuming that good news is not perfectly revealing, creating a motivate for players to delay investment once they have obtained a positive signal.

Several papers pursue related approaches to separating information production and aggregation in collective experimentation. Heidhues, Rady, and Strack (2015) finds that in a classic bandit model, outcomes improve when payoffs are private and disclosed with delay via a cheap-talk communication channel. Our paper more severely restricts possibilities for communication, generating distinctive welfare implications from private learning. Guo and Roesler (2018) augment the model of BH with discrete negative signals, which can be signaled by irreversibly dropping out of the project. In their model these signals are perfectly revealing, so that delay in dropping out is driven by free-riding rather than social-learning concerns.

Our paper is also related to models of investment timing. One set of papers assume that players receive exogenous private signals of the state, either at time zero or dynamically. Papers in this tradition include Chamley and Gale (1994), Gul and Lundholm (1995), Chari and Kehoe (2004), Rosenberg, Solan, and Vieille (2007), and Murto and Välimäki (2011, 2013). Aghamolla and Hashimoto (2020) endogenizes the precision of a private signal received at the start of the game, but does not allow agents

[4]Formally, in their model breakthroughs are publicly observed and immediately accrue a common payoff to all players. Because breakthroughs are perfectly informative, their results would not change if breakthroughs were private and players publicly invested to collect a state-contingent payoff.

to dynamically acquire information. A second strand of the literature abstracts from private information about the project and instead assumes that investment generates public signals of the project's profitability. Papers in this tradition include Décamps and Mariotti (2004), Fajgelbaum, Schaal, and Taschereau-Dumouchel (2017), and Frick and Ishii (2020). Klein and Wagner (2019) spans the two sets of papers by endowing players with time-zero private information and assuming that investment generates further public information.

None of these papers feature a tradeoff between free-riding and investment delay, a tension which plays a key role in our model. In addition, most of these papers focus on symmetric play. One exception is Gul and Lundholm (1995), which finds that asymmetric play reduces delay and raises aggregate payoffs. In contrast, our analysis identifies a non-trivial tradeoff between the payoffs generated by symmetric and asymmetric play, which can yield higher aggregate payoffs for either type of equilibrium depending on model parameters.

Finally, our paper shares important features with work by Ali (2018) and Campbell, Ederer, and Spinnewijn (2014). Ali (2018) endogenizes information acquisition in a model where players invest in a pre-determined sequence. It can therefore be viewed as a fixed-move-order analog to our exercise of endogenizing information acquisition when players invest flexibly. Campbell, Ederer, and Spinnewijn (2014) studies a team production problem in which production is private and separate from the decision to disclose progress. This separation is analogous to the separation of learning and information aggregation in our setting.

2 The model

Two firms have the opportunity to invest one unit of capital in a nonrival risky project of unknown quality. The project has underlying type θ and is either Good ($\theta = G$) or Bad ($\theta = B$). If $\theta = G$, each unit of capital invested in the project generates cashflows with a net present value of R, beginning at the time that unit of capital is invested; if $\theta = B$, the project generates no cashflows. We assume that $R > 1$, so that each unit of capital invested in the project generates positive returns in the Good state. Each firm is free to invest in the project at any time $t \in \mathbb{R}_+$. Firms are risk-neutral with common discount rate $r > 0$. Capital is indivisible, investment in the project is irreversible, and project outcomes are observed only by players who invest.[5]

Both firms begin with a common prior belief $\pi_0 \in (0,1)$ that the project is Good. Each firm $i = 1,2$ can exert costly effort to privately search for an informative signal about the project's quality, an activity we will refer to as *prospecting*. The goal of prospecting is to uncover a binary signal $S_i \in \{H, L\}$, i.e., High or Low, which is correlated with the state of the project: $\Pr(S_i = H \mid \theta = G) = q^H$ and $\Pr(S_i = L \mid \theta = B) = q^L$, with $q^H, q^L \in (1/2, 1)$. Any prospecting that a firm undertakes, and any signal that results, are observed only by the firm conducting the prospecting. Each firm can obtain at most one signal, and firms observe conditionally i.i.d. signals.

Prospecting is a dynamic process unfolding in continuous time. Over every time interval $[t, t+dt]$, each firm i chooses a prospecting rate $\lambda_t^i \geq 0$, which causes a signal to arrive with probability $\lambda_t^i \, dt$ while incurring an effort expense of $C(\lambda_t^i) \, dt$. Following much of the literature on collective experimentation,[6] we assume a linear cost structure:

$$C(\lambda) = \begin{cases} c\lambda, & \lambda \in [0, \overline{\lambda}] \\ \infty, & \lambda \in (\overline{\lambda}, \infty) \end{cases}$$

for some constant marginal cost $c > 0$ and maximum prospecting rate $\overline{\lambda}$, both of which are symmetric across firms. Conditional on prospecting rates, signal arrival times are independent across firms and independent of the state of the project.

Firms cannot observe each other's signals or prospecting intensities, nor can they observe whether another firm has received a signal or obtained a good outcome from investment. There are also no communication channels between firms. However, all investment decisions are public, introducing a channel for social learning.

2.1 Notation and assumptions

We will denote the posterior beliefs induced by one or more signals as follows: π_+ and π_{++} are the posteriors induced by one and two High signals, respectively; π_- and π_{--} are the posteriors induced by one and two Low signals; and π_{+-} is the posterior induced by one High and one Low signal. (Exchangeability implies that posterior beliefs are independent of the order of receipt of signals.) Given that High signals are

[5] A natural interpretation of the private observability of outcomes is that the project's cashflows are realized far in the future.

[6] See Keller, Rady, and Cripps (2005) and BH for classic examples of team experimentation models assuming linear experimentation costs.

more likely when the state is Good, and conversely for Low signals when the state is Bad, $\pi_{++} > \pi_+ > \pi_0, \pi_{+-} > \pi_- > \pi_{--}$. Note that in general $\pi_{+-} \neq \pi_0$, except in the special case when $q^H = q^L$.

Suppose that a firm receives a signal when its current beliefs that $\theta = G$ are $\mu \in [0, 1]$. Then the total probability that the signal is High is $h(\mu) \equiv q^H \mu + (1-q^L)(1-\mu)$, while the corresponding probability that the signal is Low is $l(\mu) \equiv 1 - h(\mu)$. The quantities $h(\mu)$ and $l(\mu)$ are the transition probabilities that a firm's posterior belief jumps up or down upon receiving a signal. Following acquisition of a signal, we will write $\mu_+ \equiv q^H \mu / h(\mu)$ for the firm's updated belief if the signal is High, and $\mu_- \equiv (1 - q^H)\mu / l(\mu)$ if the signal is low.

We impose several bounds on prospecting costs and the payoff of a Good project.

Assumption 1. $1/\pi_+ < R < 1/\pi_0$.

Under this assumption, investment in the project is ex ante unprofitable, but becomes profitable conditional on observation of a High signal.[7]

Assumption 2. $R < 1/\pi_{+-}$.

This assumption ensures that learning another firm's signal is useful even after acquisition of a High signal, since an additional Low signal would push beliefs back below the breakeven threshold. This assumption in conjunction with Assumption 1 rules out perfectly informative good news, as such signals correspond to $\pi_+ = \pi_{+-} = 1$, in which case no R can simultaneously satisfy the bounds in both assumptions. These assumptions therefore distinguish our setting from classic experimentation models like Keller, Rady, and Cripps (2005) and BH, where a single positive outcome is definitive.

Assumption 3. $c \leq \bar{c} \equiv h(\pi_+)(\pi_{++}R - 1) - (\pi_+ R - 1)$.

This assumption ensures that a second signal is at least potentially profitable to acquire, in the sense that if it could be attained instantaneously, it would provide enough information to be worth the cost. As with Assumption 2, this assumption focuses our analysis on environments in which combining information from multiple signals is strategically relevant. Note that Assumptions 1 and 2 ensure that $\bar{c} > 0$.

[7] The case $R < 1/\pi_+$ is uninteresting, as the unique equilibrium involves no prospecting and no investment by either firm.

2.2 Single-player benchmark

Consider a single firm prospecting and investing on its own, shutting down the social learning channel of our model. The firm's initial beliefs that the project is good will be taken to be $\mu < 1/R$. We will refer to this benchmark setting as *autarky*.

So long as the firm has acquired no signal, it learns nothing about the project and its beliefs remain fixed at μ. An optimal prospecting strategy is therefore stationary. This behavior differs from the cutoff strategies which are optimal when learning from Poisson bandits, for instance as in BH. In Poisson bandit models, lack of arrival of a signal is itself news about the underlying state, leading to belief updating. In our model, by contrast, lack of signal acquisition does not signal anything, positive or negative, about the true project state; no news truly is no news until a signal arrives.[8] Once the firm has acquired a signal, no further information is available. It then faces a simple static choice of whether or not to invest, which it resolves by comparing its posterior beliefs to the investment threshold $1/R$.

The optimal prospecting strategy depends on whether the firm's initial beliefs μ lie above a critical threshold, which we will denote π_A and refer to as the *autarky threshold*. It is formally characterized as the unique belief satisfying $h(\pi_A)(\pi_{A+}R - 1) = c$, which equalizes the marginal flow gains and costs from a unit of prospecting. (Recall our notational convention that $\pi_{A+} = q^H \pi_A / h(\pi_A)$ are the posterior beliefs following receipt of a High signal, when beliefs are π_A prior to observing the signal.) If $\mu < \pi_A$, the firm abandons prospecting immediately. On the other hand, if $\mu > \pi_A$, then the firm prospects at the maximum rate $\overline{\lambda}$ until a signal is acquired. Note that π_A is increasing in the cost parameter c.

Abandonment of prospecting if beliefs fall below π_A occurs even with multiple firms: If π_0 lies below π_A, then no prospecting or investing takes place in equilibrium, despite the potential for social learning. Going forward we will assume that $\pi_0 > \pi_A$ for all costs below $\overline{c}$, which the following lemma establishes is equivalent to assuming R is sufficiently large.

Lemma 1. *There exists a unique* $R_0 \in (1/\pi_+, 1/\max\{\pi_{+-}, \pi_0\})$ *such that* $\pi_0 > \pi_A$ *for every* $c \leq \overline{c}$ *if and only if* $R > R_0$.

Assumption 4. $R > R_0$.

[8]A similar signal acquisition technology is employed in Akcigit and Liu (2016).

Lemma 1 ensures that this bound is compatible with the restrictions on R imposed by Assumptions 1 and 2. The bound could be dispensed with, at the cost of a more stringent upper bound on allowed prospecting costs. To streamline our analysis, we maintain Assumption 4 going forward.

3 Equilibrium analysis

In this section we characterize the set of perfect Bayesian equilibria of the model. Going forward, we will use the term *equilibrium* without qualification to refer to elements of this set. We find that our model has exactly three equilibria. One equilibrium is symmetric and exhibits no free-riding or investment delay, but leads both firms to eventually abandon prospecting for information about the project. The remaining "leader-follower" equilibria feature distinct roles for the two firms, with one firm who takes the lead in prospecting and investing while the other firm plays a passive follower role. In general this equilibrium features either free-riding, investment delay, or both by the follower, with the mix shifting from investment delay toward free-riding as prospecting costs rise.

The section is structured as follows. In Section 3.1, we describe each firm's optimal continuation strategy after observing investment by the other firm. In Sections 3.2 and 3.3, we characterize the symmetric and leader-follower equilibria and provide intuition for their properties. In Section 3.4, we prove that no other equilibria exist.

3.1 Behavior after observing investment

In the spirit of backward induction, we first characterize a firm's optimal continuation strategy after observing the other firm invest. It can be shown that in any equilibrium, the first firm to invest is always in possession of a High signal.[9] The remaining firm therefore finds itself in a stationary single-player environment analogous to the autarky benchmark studied in Section 2.2. If the firm has already acquired a signal, its beliefs are either $\pi_{++} > 1/R$ or $\pi_{+-} < 1/R$, and no further information can be acquired. The firm therefore either invests immediately if its signal is High, and abandons the project otherwise.

On the other hand, if the firm has not yet acquired a signal, its beliefs are $\pi_{+} >$

[9] See Appendix A for a formal derivation of this result.

$1/R$ and it has the opportunity to acquire a signal before investing. This additional signal is pivotal, given that $\pi_{+-} < 1/R$, and would be worth the cost of acquiring if not for time discounting, given $c \leq \bar{c}$. Whether the signal is in fact worth acquiring depends on the comparison between the gains from more information, mediated by R, and the cost and delay of obtaining it, captured by c, r, and $\overline{\lambda}$.

As R rises, the firm becomes less willing to acquire an additional signal, because the downside of a bad project becomes less important relative to the upside of a good one. The firm also becomes less willing to wait if prospecting or delay costs rise, i.e., if c or r increase or $\overline{\lambda}$ decreases. Either acquiring an additional signal or investing immediately can be optimal, depending on parameters. The following lemma formally states how the optimal strategy changes with the discount rate r, a comparative static that will be particularly useful for later results.[10] (The proof is straightforward, and so is omitted for brevity.)

Lemma 2. *There exists a threshold discount rate $r^* \geq 0$ such that in any equilibrium, subsequent to investment by some firm:*

- *If $r \leq r^*$, the remaining firm prospects at rate $\overline{\lambda}$ until acquiring a signal, and invests immediately if it acquires a High signal.*
- *If $r > r^*$, the remaining firm invests immediately if it has not yet acquired a Low signal.*

3.2 The symmetric equilibrium

We now characterize the unique symmetric equilibrium of the model. This equilibrium exhibits no free-riding or investment delay, but does involve eventual abandonment of prospecting by both firms.

To state the equilibrium, we define a time threshold at which a firm's posterior beliefs reach π_A, assuming the other firm never delays signal acquisition or investment. Suppose that some firm i prospects at rate $\overline{\lambda}$ forever and invests immediately whenever it obtains a High signal. Let $\mu^{\overline{\lambda}}(t)$ denote the associated posterior beliefs of firm $-i$ that $\theta = G$, conditional on observing no investment by firm i until time t. These beliefs decline over time, converging to π_- as $t \to \infty$ and firm $-i$ becomes sure that

[10] An identical result holds with respect to $1/\overline{\lambda}$. Analogous results could also be stated for R and c, but with some additional care needed to account for the boundary conditions on these parameters.

continued lack of investment implies that firm i has obtained a Low signal. Since $\pi_{+-} < 1/R$, it must be that $\pi_A > \pi_-$, and so $\mu^{\overline{\lambda}}(t)$ crosses the autarky threshold π_A at some finite time, which we will denote $T^A \equiv (\mu^{\overline{\lambda}})^{-1}(\pi_A)$.

Proposition 1 (The symmetric equilibrium)**.** *There exists a symmetric equilibrium in which, whenever no investment has occurred:*

- *If firm $i \in \{1, 2\}$ has not obtained a signal, it prospects at rate*

$$\lambda_t^i = \begin{cases} \overline{\lambda}, & t \leq T^A \\ 0, & t > T^A \end{cases}$$

- *If firm $i \in \{1, 2\}$ has obtained a High signal, it invests immediately.*

This equilibrium unfolds as follows. Absent observing investment by the other firm, each firm prospects at rate $\overline{\lambda}$ until time T^A. Afterward each firm stops prospecting forever. If at any time a firm observes investment (before or after time T^A), it follows the optimal continuation strategy characterized in Lemma 2. And if at any time a firm is in possession of a High signal (on or off the equilibrium path), it invests immediately. Finally, no firm invests while in possession of no signal or a Low signal.

One key feature of this equilibrium is that it exhibits *neither free-riding nor investment delay*. That is, at no point in time does a firm stop prospecting while its beliefs are above π_A, nor does any firm in possession of a High signal ever wait to invest. Another important feature is that *both firms eventually abandon prospecting for information about the project*. If by time T^A no firm has invested, both firms cease efforts to acquire a signal forever afterward.[11]

The prospecting strategies arising in this equilibrium do not reflect unique best replies for either firm. Indeed, subsequent to time T^A, each firm is indifferent between prospecting or not, as their beliefs remain fixed at π_A forever afterward. However, abandoning prospecting is the unique continuation outcome that can be sustained as part of an equilibrium. For it is precisely the lack of information arriving after beliefs reach π_A which makes it optimal for firms to prospect at all times prior to T^A. If some

[11]This effect resembles the investment collapse phenomenon described in Chamley and Gale (1994). In that paper a collapse is precipitated by randomization over investment at early stages of the game. By contrast, in our setting abandonment of prospecting is induced by stochastic information acquisition.

firm were to continue prospecting, they would drive the other firm's beliefs below π_A in finite time, and that firm would then no longer optimally prospect until time T^A.

In the remainder of this subsection, we provide some intuition for the optimality of each firm's strategy in this equilibrium. First consider each firm's investment strategy. Because both firms quit prospecting at time T^A, each firm's beliefs are always at least π_A for all times, even absent a signal or any observed investment. This means that after obtaining a High signal, each firm's beliefs lie above $1/R$ forever. So by waiting to invest, no firm ever obtains enough negative information to change their optimal investment decision, meaning any delay in investing is suboptimal.

Now consider the equilibrium prospecting rule. At all times prior to T^A, each firm's beliefs are above the autarky level, and so they would optimally prospect in a single-player environment. However, in the presence of social learning, each firm faces a tradeoff between acquiring information today in order to invest more quickly, or saving on prospecting costs by waiting to see whether the other firm acts. Early on, beliefs about the project are relatively good and the value of prospecting exceeds the value of waiting. However, as firms approach time T^A, both values converge to zero, since both the single-player returns to prospecting as well as the probability that the other firm eventually invests vanish.

The comparison between the value of prospecting and waiting at times close to T^A is mediated by the cost of prospecting. To see this, consider firm i's choice of whether to prospect an instant before T^A. In a state of the world in which firm $-i$ does not invest by time T^A, firm i's beliefs at time $T^A - dt$ conditioning on this extra information are exactly π_A. In that case, its expected payoff at time $T^A - dt$ is zero whether or not it prospects, and the net gain from prospecting is zero.

Meanwhile, in a state of the world in which firm $-i$ does invest by time T^A, firm i's expected payoff at time $T^A - dt$ is $U_{FR} = \overline{V}$ if it free-rides, where

$$\overline{V} = \max\left\{\pi_+ R - 1, \frac{\overline{\lambda}}{\overline{\lambda} + r}\left(h(\pi_+)(\pi_{++}R - 1) - c\right)\right\}$$

is the larger of the values of investing immediately or obtaining another signal at time T^A. (Recall that the optimal continuation strategy depends on the sign of $r - r^*$, as characterized in Lemma 2.) On the other hand if it prospects, its expected payoff is

$$U_P = \overline{V}(1 - \overline{\lambda}\, dt) + h(\pi_+)(\pi_{++}R - 1)\overline{\lambda}\, dt - \overline{\lambda} c\, dt,$$

where the first term accounts for the possibility that i fails to acquire a signal by time T^A, in which case it follows its optimal continuation strategy upon seeing $-i$ invest; the second term accounts for the possibility that i acquires its own signal by time T^A, in which case it invests only if its own signal is positive; and the final term accounts for the cost of prospecting.[12] The net gain from prospecting an instant before T^A in this state of the world is therefore

$$U_P - U_{FR} = \overline{\lambda}(h(\pi_+)(\pi_{++}R - 1) - \overline{V} - c)\, dt.$$

If $r \leq r^*$, then the follower optimally acquires its own signal upon seeing the leader invest. In this case the payoff of prospecting trivially dominates the payoff of free-riding, since the follower expects to eventually acquire the signal anyway. On the other hand if $r > r^*$, then the gains from prospecting are non-negative whenever $c \leq \bar{c} = h(\pi_+)(\pi_{++}R - 1) - (\pi_+R - 1)$, as imposed in Assumption 3. This condition ensures that the improvement $h(\pi_+)(\pi_{++}R-1)-(\pi_+R-1)$ to i's investment decision from acquiring an additional signal is larger than the cost c of acquiring it in the pivotal state of the world in which i acquires a signal and sees $-i$ invest in the next period. Note that the condition $c \leq \bar{c}$ does not involve the discount rate r, since firm i is not deciding whether to delay investment after seeing $-i$ act in order to acquire a signal, but rather is deciding whether to "front-run" firm $-i$ by acquiring a signal ahead of $-i$'s own action.

This calculation makes clear that each firm's willingness to prospect until time T^A is closely linked to the information gained from a second signal. Were a single High signal close to perfectly revealing, there would be no value to prospecting just before time T^A, and free-riding would necessarily arise in equilibrium. Our assumption that signals are noisy drives the distinction between our results and those of BH, who predict free-riding in the symmetric equilibrium of a model with perfectly revealing good news.

3.3 The leader-follower equilibrium

We next characterize a pair of asymmetric equilibria in which firms adopt distinct leader and follower roles. Unlike the symmetric equilibrium, prospecting is never

[12] Our expressions for U_{FR} and U_P drop terms related to discounting, which are second-order in dt when computing $U_P - U_{FR}$.

abandoned entirely: with probability 1 at least one firm eventually acquires a signal about the project. However, they feature free-riding, investment delay, or both by the follower. Throughout this section, we describe the equilibrium in which firm 1 is the leader and firm 2 is the follower. By symmetry, another equilibrium exists with the roles of the firms reversed.

Proposition 2 (The leader-follower equilibrium)**.** *There exists an equilibrium in which, whenever no investment has occurred:*

- *If firm* 1 *has not obtained a signal, it prospects at rate* $\lambda_t^1 = \overline{\lambda}$.
- *If firm* 1 *has obtained a High signal, it invests immediately.*

This equilibrium is characterized by time thresholds $\overline{T}_F, T_F^* \in [0, \infty)$ *such that, whenever no investment has occurred:*

- *If firm* 2 *has not obtained a signal, it prospects at rate*

$$\lambda_t^2 = \begin{cases} \overline{\lambda}, & t \leq \overline{T}_F \\ 0, & t > \overline{T}_F \end{cases}$$

- *If firm* 2 *has obtained a High signal, it invests immediately if* $t \leq T_F^*$*, and waits to invest otherwise.*

These time thresholds are uniquely determined and satisfy $\min\{\overline{T}_F, T_F^*\} < T^A$.

This equilibrium unfolds as follows. If either firm observes investment when not in possession of a signal, it follows the optimal continuation strategy characterized in Lemma 2. Prior to such an event, the leader prospects at rate $\overline{\lambda}$ until it obtains a signal. If at any time the leader is in possession of a High signal (on or off the equilibrium path), it invests immediately. Meanwhile, the follower prospects only up until the threshold time $\overline{T}_F < \infty$. If at any time t the follower is in possession of a High signal (on or off the equilibrium path), it invests immediately if $t < T_F^*$, and otherwise it waits for action by the leader. The bound $\min\{\overline{T}_F, T_F^*\} < T^A$ implies that the follower becomes passive, i.e., ceases investing ahead of the leader on the equilibrium path, earlier than it would have in the symmetric equilibrium.[13]

[13]Note that when $T_F^* > \overline{T}_F$, firm 2 does not acquire a signal on the equilibrium path at times $[\overline{T}_F, T_F^*]$ unless firm 1 invests. Nonetheless, the threshold time T_F^* is the unique continuation strategy consistent with the requirements of perfect Bayesian equilibrium in the off-path information sets in which firm 2 has acquired a signal.

Unlike the symmetric equilibrium of Proposition 1, the leader-follower equilibrium exhibits free-riding, investment delay, or a combination of the two. The following lemma characterizes when each arises as a function of the cost of prospecting.

Lemma 3 (Comparison of thresholds)**.** *There exist cost thresholds c^*, c_* satisfying $\overline{c} \geq c^* > c_* > 0$ such that:*

- *If $c > c^*$, then $\overline{T}_F \leq T_F^*$, while if $c < c^*$, then $\overline{T}_F > T_F^*$.*
- *If $c > c_*$, then $\overline{T}_F < T^A$, while if $c < c_*$, then $\overline{T}_F > T^A$.*

Further, if r is sufficiently small, then $\overline{c} > c^*$.

This lemma establishes that equilibrium behavior moves through three distinct regimes as prospecting costs rise. When costs are below c_*, the follower delays investment but never free-rides; that is, it prospects at least until its beliefs fall below the autarky threshold. Meanwhile when costs are between c_* and c^*, the follower initially delays investment and eventually free-rides. Finally, when costs are above c^*, the follower free-rides but never delays investment.[14] Thus as prospecting costs rise, the mix of free-riding and investment delay shifts toward the former and away from the latter.

The findings of Lemma 3 are depicted graphically in Figure 1. This diagram describes the follower's equilibrium strategy as a function of time (on the horizontal axis) and the cost of prospecting (on the vertical axis). In region I, the follower prospects and invests immediately upon acquisition of a positive signal.[15] In region II, the follower free-rides and does not obtain a signal on the equilibrium path. This region exists if costs are sufficiently high; in particular, above the cost threshold c_* defined by $\overline{T}_F = T^A$.[16] In regions III and IV, the follower prospects but delays investment if it obtains a positive signal. This region exists if costs are sufficiently low, i.e., below the cost threshold c^* defined by $T_F^* = \overline{T}_F$. In region IV, the follower's prospecting would be unprofitable in the autarky benchmark, a phenomenon we analyze further below. This region is present whenever costs are below c_*.

[14] While the firm does eventually stop investing when in possession of a High signal, any such signal must have been acquired due to a deviation from its equilibrium strategy. We describe a firm's strategy as exhibiting delay only when waiting arises on-path, consistent with the convention used in the investment timing literature. (See, e.g., Chamley and Gale (1994).)

[15] This region need not exist for all model parameterizations. In particular, $T_F^* = 0$ if r is sufficiently small.

[16] This region also disappears if c is sufficiently large that $T^A = 0$. However, Assumption 4 ensures that $T^A > 0$ for all $c \leq \overline{c}$.

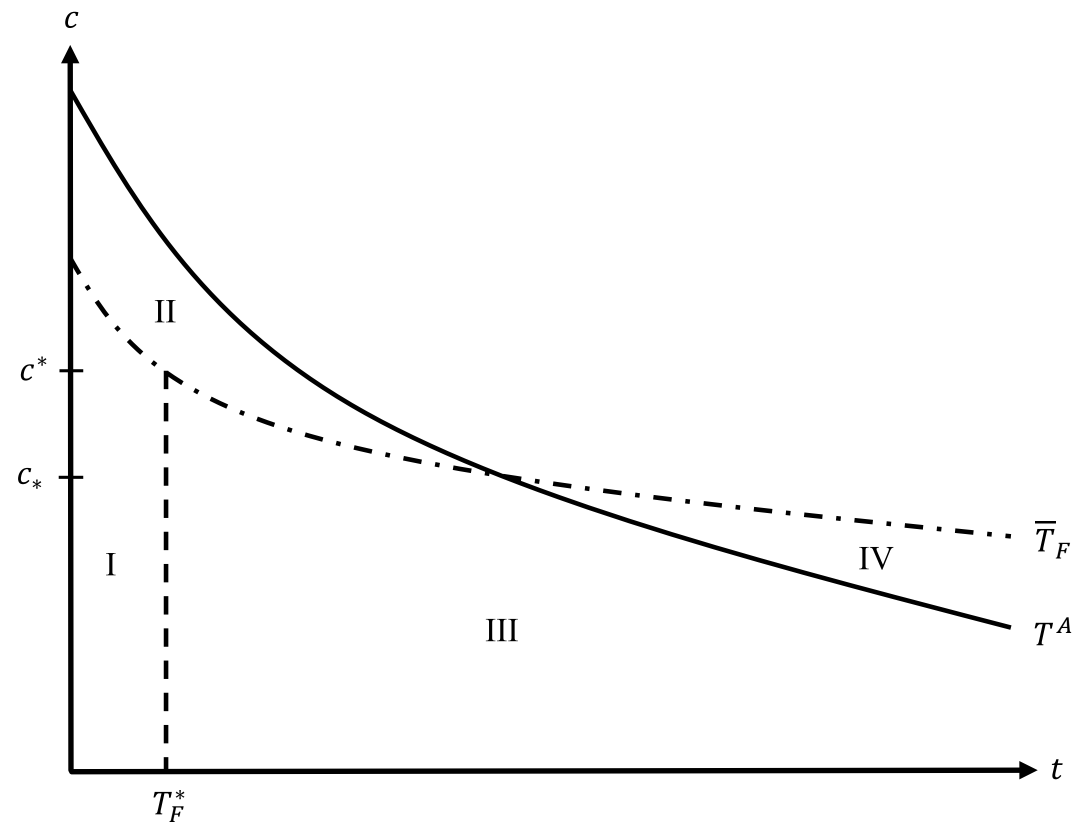

Figure 1: The follower's strategy as a function of time (horizontal axis) and c (vertical axis). In region I it remains active. In region II it free-rides. In regions III and IV it prospects but delays investment. In region IV prospecting is unprofitable in the autarky benchmark.

In the remainder of this subsection, we provide some intuition for the structure of the leader-follower equilibrium. Each firm's behavior can be understood by comparing their incentives here and in the symmetric equilibrium. In the latter equilibrium, social learning stops at time T^A for each firm, providing incentives which are just strong enough to remain active up to this time. By contrast, in the current setting the leader remains active longer, weakening the follower's incentives to prospect and invest. Hence the follower must become passive prior to time T^A, as explained in more detail below. This change in behavior in turn reinforces the leader's desire to remain active, since it no longer learns enough from the follower's activities to push its beliefs below the autarky threshold. Hence the follower's behavior induces the leader to actively prospect and invest at all times.

Given the opportunity for social learning provided by the leader's continued activity after time T^A, at all times prior to T^A the follower's continuation value must be strictly positive. But since the return to actively prospecting and investing approaches zero for times close to T^A, it cannot be optimal for the follower to continue both of these activities until T^A. In other words, by time T^A at least one of A) the follower's value of actively prospecting, or B) his value of investing following acquisition of a signal, must fall below the option value of waiting to observe investment by the leader. The time at which event A occurs is precisely $\overline{T}_F$, while the time of event B is T_F^*.

It is not necessarily true that the value of prospecting is exhausted before the value of investing, because obtaining a signal preemptively eliminates the delay involved in obtaining a signal after seeing the leader invest. This "front-running" motive yields a non-zero benefit from obtaining a signal even after T_F^*. The comparison between T_F^* and $\overline{T}_F$ hinges on the cost of prospecting. Intuitively, the follower's decision to delay investment is independent of c, since waiting to invest does not involve any expenditure of prospecting costs. By contrast, the more costly prospecting becomes, the sooner the follower prefers to free-ride. Hence for low c the follower begins delaying investment before it stops prospecting, while for high c the opposite is true. Strikingly, when $c < c_*$ the follower prospects even after its beliefs fall below the autarky threshold. In this regime, the front-running motive boosts the value of acquiring a signal compared to a one-player environment, encouraging additional prospecting.

3.4 Characterization of the equilibrium set

So far we have demonstrated the existence of three equilibria: a symmetric equilibrium and two leader-follower equilibria (which are identical up to permutation of firms). We now establish that these equilibria constitute the entire equilibrium set.[17]

Proposition 3. *There exist no equilibria, in pure or mixed strategies, beyond those characterized in Propositions 1 and 2.*

The bulk of the proof involves showing that, up to some technicalities, all equilibria must be in strategies analogous to those arising in Propositions 1 and 2: each firm

[17] More precisely, the proposition establishes essential uniqueness, up to the usual continuous-time degeneracies on sets of times and states of measure zero.

$i \in \{1, 2\}$ prospects at rate $\overline{\lambda}$ up to some threshold time $\overline{T}_i$ and then stops prospecting afterward; and similarly, if it has received a High signal, it invests immediately up to a threshold time T_i^* and then waits to invest afterward. We call strategies of this form *threshold strategies*.

The optimality of a threshold investing rule relies on an argument ruling out waiting for a (possibly random) period and then investing. Such a strategy would merely delay investment without conditioning it on the arrival of information in any useful way. Therefore once it becomes optimal to wait at all, any optimal strategy must involve waiting until the other firm has invested. The optimality of a threshold prospecting rule is more technical, and requires studying the dynamics of the HJB equation. Essentially, the proof establishes that the moment free-riding becomes even weakly optimal, a firm's value function must evolve in such a way that free-riding remains strictly optimal forever afterward. As suggested by the discussion following Proposition 1, the cost bound $c \leq \overline{c}$ plays a key role in this argument.

Within the class of equilibria in threshold strategies, the equilibrium set can be narrowed down by a straightforward classification argument. The symmetric equilibrium can be characterized as the unique equilibrium in which both firms stop investing on-path at the same time. Within this class, the only way that both firms can become passive at the same time in equilibrium is if both firms' beliefs reach π_A at this time. For if some firm's terminal beliefs were any higher, that firm would prefer to continue prospecting and investing afterward, and if its beliefs were any lower, it would prefer to become passive sooner. Backward induction then pins down the symmetric equilibrium as the unique behavior consistent with this outcome.

The leader-follower equilibrium can be characterized as the unique equilibrium in the remaining case that some firm i remains active, that is, invests along the equilibrium path, longer than the other. Let $\widehat{T}_{-i}$ be the time at which firm $-i$ becomes passive, and call firm $-i$ the follower. In this case firm i, the leader, is effectively in autarky after time $\widehat{T}_{-i}$ and prospects and invests immediately at all future times. To sustain an equilibrium, it must then be a best response to the leader's continuation strategy for the follower to stop investing on-path at time $\widehat{T}_{-i}$. This optimality condition uniquely pins down $\widehat{T}_{-i}$, which may be the time at which the follower either stops prospecting or stops investing, depending on model parameters. Once this time is pinned down, it can be shown that the leader's unique best response is to remain active prior to time $\widehat{T}_{-i}$, which uniquely determines the remainder of the equilibrium.

4 Comparing equilibrium payoffs

We have seen that our model has exactly two distinct equilibrium structures. In this section we compare individual and aggregate payoffs across equilibria.

Let V^S be the expected payoff of each firm in the symmetric equilibrium, and let V^L and V^F be the expected payoffs to the leader and follower, respectively, in the leader-follower equilibrium. Aggregate payoffs in the symmetric equilibrium are then $2V^S$, while in the leader-follower equilibrium they are $V^L + V^F$. The following proposition examines how both individual and aggregate payoffs compare across the two equilibria.

Proposition 4. *$V^F > V^S \geq V^L$. If r is sufficiently small, then $V^L + V^F > 2V^S$. There exists a $\underline{c} < \bar{c}$ (independent of r) such that if $c > \underline{c}$, then $2V^S > V^L + V^F$ for r sufficiently large.*

The first result of the proposition is that the symmetric equilibrium generates lower payoffs for each firm than the follower's payoff, but (weakly) higher payoffs than the leader's payoff. Intuitively, the leader-follower equilibrium generates more information from the leader but less from the follower than each firm would produce in the symmetric equilibrium, and this change in social learning is reflected in the remaining firm's payoff.

Interestingly, the leader is not necessarily strictly worse off than it would be in the symmetric equilibrium. This is because when the discount rate is low, each firm acquires its own signal before investing, even after seeing the other firm invest. (See Lemma 2.) In addition, in the symmetric equilibrium neither firm waits for the other to invest once they are in possession of a High signal. Social learning therefore turns out not to be pivotal for investment in the symmetric equilibrium at low discount rates, and so the additional information generated does not raise payoffs.

This second result of the proposition is that when firms are patient, the leader-follower equilibrium yields higher total firm profits than the symmetric one, while when firms are impatient (and costs aren't too low) the symmetric equilibrium is superior. If the discount rate is low enough that $V^L = V^S$, this result is an immediate consequence of the individual payoff ranking. However, if $V^L < V^S$, comparing aggregate payoffs requires a balancing of higher payoffs generated by the symmetric equilibrium early on, against higher payoffs generated by the leader-follower equilibrium later. Consequently, the comparison between the two equilibria turns on the discount rate.

More precisely, in the symmetric equilibrium both firms contribute to social learning by actively prospecting and investing until the time T^A. By contrast, in the leader-follower equilibrium the leader remains active forever, while the follower remains active only up to some time $\widehat{T}_F < T^A$. Comparing welfare therefore amounts to comparing aggregate social learning in each equilibrium, taking into account time discounting.

In the symmetric equilibrium, more social learning occurs during the time interval $[\widehat{T}_F, T^A]$ than in the leader-follower equilibrium, while the latter equilibrium features more social learning during the time interval $[T^A, \infty)$. When firms are patient, the long duration of social learning in the leader-follower equilibrium is the most important factor determining welfare, and so aggregate payoffs are higher in this equilibrium. By contrast, when firms are impatient, the additional social learning generated early on in the symmetric equilibrium becomes important.

The subtlety in this argument is that $\widehat{T}_F$ approaches T^A as the discount rate goes to zero, requiring a careful calculation of total gains in the limit as $r \to 0$. It turns out that, under an appropriate normalization, the limiting gains from additional social learning early in the symmetric equilibrium are strictly positive and increasing in c. Thus when r is small and c is sufficiently large, these normalized gains outweigh the normalized losses from reduced social learning later on, yielding higher aggregate payoffs than in the leader-follower equilibrium.[18]

The ambiguity of this payoff comparison stands in contrast to the findings of previous work on investment timing and collective experimentation. A classic example in the investment timing literature is Gul and Lundholm (1995), who find that asymmetric equilibria eliminate the war of attrition inherent in symmetric play and reveal private information more quickly, improving aggregate welfare. By contrast, since information acquisition is endogenous in our model, asymmetric play generates an additional profit loss by reducing the incentives for the second mover to produce and reveal information. The relative performance of symmetric and asymmetric play

[18]The cost bound $\underline{c}$ need not be very stringent. In particular, it can be shown that $\underline{c} < 0$ when positive and negative signals are both sufficiently informative.

then becomes a horse race between the war of attrition effect of the symmetric equilibrium, and the free-riding of the asymmetric equilibrium. This finding demonstrates the importance of modeling incentives for information acquisition alongside investment timing when both effects are present in applications.

Meanwhile in the collective experimentation literature, BH finds that asymmetric play increases aggregate payoffs versus symmetric play. Specifically, in the 2-player version of their model, they characterize a continuum of asymmetric equilibria indexed by the time at which the follower stops free-riding and begins exerting effort. They show that aggregate payoffs are increasing in the amount of time the follower spends free-riding. Key to their result is the fact that the more players are actively exerting effort, the less total effort is exerted. By contrast, in our setting asymmetric play has an ambiguous effect on total effort: effort early on is lower than in the symmetric equilibrium while effort at later times is higher. Our contrasting welfare results are therefore driven by important differences in behavior across the two models in both symmetric and asymmetric equilibria.

5 Conclusion

We study a model of strategic investment timing with endogenous information acquisition, with the aim of understanding the interplay of incentives for free-riding and investment delay. We find that the extent and mix of free-riding and investment delay varies across equilibria as well as with the cost of acquiring information. We further find that the equilibrium which maximizes aggregate payoffs varies with model parameters, in particular the discount rate. These results are all closely linked to our central assumption that positive signals are imperfectly informative, revealing new economic forces that are absent in models of strategic experimentation with observable or perfectly revealing signals.

One limitation of our current analysis is its focus on a two-player setting. Extending our work to accommodate many players would bring it closer to applications, as well as permit a richer study of the possibilities of asymmetric play. In particular, with many players there might exist additional asymmetric equilibria featuring multiple active players. Comparing aggregate payoffs across different asymmetric configurations could reveal novel tradeoffs that further illuminate when and how asymmetric play boosts payoffs.

Our analysis also restricts attention to environments in which investment represents a pure information externality. In some applications, investment may additionally generate payoff externalities. For instance, early-stage startups may exhibit increasing returns to scale and generate higher profits, or a greater probability of success, when they are better-funded. In that case, investment by one firm would raise the return on investment by another, strengthening incentives for strategic delay. Extending our model to incorporate increasing returns to scale would enhance its realism in such applications and allow informational and payoff externalities to be compared as sources of strategic delay.

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Appendices

A Regular strategies

In this appendix we establish that in any perfect Bayesian equilibrium, lack of investment is (weakly) bad news about the state, while investment indicates that the other firm has received a High signal.

Definition A.1. *A firm's strategy is* regular *if:*

- *Investment never occurs after receipt of a Low signal,*
- *Investment without a signal occurs only in histories in which the other firm has invested.*

The following lemma establishes that all firms choose regular strategies in equilibrium. We shall invoke this fact repeatedly in what follows to focus our analysis on a firm's best response to play of a regular strategy by his rival.

Lemma A.1. *In any equilibrium, each firm's strategy is regular.*

Proof. Fix an equilibrium. First consider a firm who has obtained a Low signal. Then regardless of his beliefs about the content of any signal obtained by the other firm, his posterior belief that the state is Good cannot be higher than π_{+-}. As $\pi_{+-}R - 1 < 0$ by assumption, investment in such a history is unprofitable. Thus in any equilibrium, no firm invests in such a history.

Now consider a firm i who has obtained no signal by time t. If i believes that $-i$, when following its equilibrium strategy, would have invested with probability strictly less than 1 by time t, then this history is on-path. Firm i may then use Bayes' rule to update its beliefs about firm $-i$'s signal, and as $-i$ does not invest when in receipt of a Low signal, lack of investment by time t is weakly negative news about S_{-i} and therefore about θ. Thus firm i's posterior beliefs that $\theta = G$ are no higher than π_0, and investment in such a history is unprofitable. Thus in any equilibrium, no firm invests in such a history.

The remaining possibility is that firm i has obtained no signal and is in a history at which firm $-i$'s strategy called for investment with probability 1 prior to time t. Such histories are off-path, and firm i is then free to choose its beliefs about S_{-i} arbitrarily. To complete the proof, we argue that such off-path histories cannot arise in any equilibrium. Let $\rho^i(t)$ be the cumulative time-t probability that each firm, under its equilibrium strategy, invests prior to time t absent observing investment by its rival. Each ρ^i is weakly increasing and left-continuous. Off-path histories correspond to $\rho^i(t) = 1$.

Let $t^* \equiv \inf\{t \ : \ \max\{\rho^1(t), \rho^2(t)\} = 1\}$, and suppose by way of contradiction that $t^* < \infty$. Prior to time t^*, histories are on-path, and so no firm invests when in possession of no or a Low signal. As a result, it must be that $\rho^i(t) \leq (1 - \exp(-\overline{\lambda}t))h(\pi_0)$ for each $t < t^*$ and firm i, since each firm's prospecting rate is bounded above by $\overline{\lambda}$. Thus by left-continuity, $\rho^1(t^*), \rho^2(t^*) < 1$. Then also at time t^*, histories involving lack of investment are on-path for both firms, meaning no firm invests when in possession of a Low signal. But also by definition there exists a firm i for which $\rho^i(t) = 1$ for arbitrarily small $t > t^*$, meaning that firm i must invest with strictly positive probability when in possession of no signal at time t^*. This is a contradiction, and so $t^* = \infty$, meaning all histories are on-path. □

B Belief updating identities

In this appendix we derive several useful identities involving posterior beliefs about the state in the event no investment by the other firm has been observed.

Fix a firm i and a strategy for firm $-i$ which is regular (as defined in Appendix A) with non-random prospecting and a threshold investment policy. We will let $\mu^i(t)$ denote firm i's time-t belief that the state is Good, supposing it has obtained no signal and observed no investment. And we will let $\nu^i(t)$ denote firm i's time-t belief that firm $-i$ has not yet obtained a signal, given that it has not yet invested.

Lemma B.1. *$\mu^i(t)$ is absolutely continuous, and*

$$\dot{\mu}^i(t) = -\mathbf{1}\{t < T^*_{-i}\}\nu^i(t)\lambda^{-i}(t)h(\pi_0)(\pi_+ - \mu^i(t))$$

almost everywhere.

Proof. For all times $t > T^*_{-i}$, firm i is in autarky with fixed beliefs, in which case $\mu^i(t)$ is trivially absolutely continuous and satisfies the stated identity. So consider times $t \leq T^*_{-i}$. Then firm $-i$ invests at variable Poisson rate $\nu^i(t)\lambda^{-i}(t)h(\pi_0)$, and arrival of investment causes beliefs to jump from $\mu^i(t)$ to π_+. It follows that $\mu^i(t)$ is absolutely continuous and satisfies the Bayes' rule condition that the average rate of change of beliefs must be zero:

$$\nu^i(t)\lambda^{-i}(t)h(\pi_0)(\pi_+ - \mu^i(t)) + \dot{\mu}^i(t) = 0,$$

which is the desired identity. □

Lemma B.2.

$$\dot{\mu}^i(t) = -\mathbf{1}\{t < T^*_{-i}\}\lambda^{-i}(t)\frac{\mu^i(t) - \pi_-}{\pi_+ - \pi_-}(\pi_+ - \mu^i(t))$$

almost everywhere.

Proof. For all times $t \geq T^*_{-i}$, firm i is in autarky with fixed beliefs, in which case the identity trivially holds. So assume $t < T^*_{-i}$. Define

$$\Omega^{-i}(t) = \exp\left(-\int_0^t \lambda^{-i}(s)\,ds\right).$$

to be the cumulative probability that firm $-i$ has not obtained a signal by time t. By Bayes' rule

$$\mu^i(t) = \frac{(\Omega^{-i}(t) + (1 - \Omega^{-i}(t))(1 - q^H))\pi_0}{\Omega^{-i}(t) + (1 - \Omega^{-i}(t))l(\pi_0)} = \frac{\Omega^{-i}(t)\pi_0 + (1 - \Omega^{-i}(t))l(\pi_0)\pi_-}{\Omega^{-i}(t) + (1 - \Omega^{-i}(t))l(\pi_0)}.$$

Solving this identity for $\Omega^{-i}(t)$ yields

$$\Omega^{-i}(t) = \frac{l(\pi_0)}{h(\pi_0)}\frac{\mu^i(t) - \pi_-}{\pi_+ - \mu^i(t)}.$$

Taking the log of both sides, differentiating, and using the identity $\frac{d}{dt}\log\Omega^{-i}(t) = -\lambda^{-i}(t)$ yields the desired relationship. $\square$

C Value functions and the HJB equation

In this appendix we describe properties of the the firm's continuation value function in several important classes of histories, supposing that the other firm uses a strategy which is regular (as defined in Appendix A) with non-random prospecting and a threshold investment policy.

We will use the following notation for value functions in different histories. $V^i(t)$ will denote firm i's time-t continuation value function given no signal and no investment by firm $-i$. $\overline{V}$ will denote i's continuation value upon seeing firm $-i$ invest. (Note that $\overline{V}$ is independent of i and t.) $V^i_+(t)$ will denote firm i's time-t continuation value function given a high signal and no investment by firm $-i$. Finally, $\widetilde{V}^i(t)$ will denote firm i's expected time-t continuation value after obtaining a signal, given no investment by firm $-i$. Since obtaining a Low signal leads to no investment, it follows that $\widetilde{V}^i(t) = h(\mu^i(t))V^i_+(t)$.

Pre-signal/investment: By standard arguments, V^i is the unique bounded, absolutely continuous function satisfying the HJB equation

$$rV^i(t) = \overline{\lambda}\left(\widetilde{V}^i(t) - c - V^i(t)\right)_+ + \mathbf{1}\{t < T^*_{-i}\}\nu^i(t)\lambda^{-i}(t)h(\pi_0)(\overline{V} - V^i(t)) + \dot{V}^i(t),$$

where $\nu^i(t)$ is firm i's time-t belief that firm $-i$ has not yet obtained a signal given that it has not yet invested. Using Lemma B.1, the second term on the rhs may be rewritten in terms of $\mu^i(t)$, firm i's posterior belief that the state is Good:

$$rV^i(t) = \overline{\lambda}\left(\widetilde{V}^i(t) - c - V^i(t)\right)_+ - \frac{\dot{\mu}^i(t)}{\pi_+ - \mu^i(t)}(\overline{V} - V^i(t)) + \dot{V}^i(t).$$

Note that the sign of $\widetilde{V}^i(t) - c - V^i(t)$ determines firm i's optimal prospecting rule: When it is strictly positive, the firm optimally prospects at rate $\overline{\lambda}$; when it is strictly negative, the firm optimally prospects at rate 0; and when it is zero, any prospecting rate is optimal.

At various points in our analysis, it will be useful to express the HJB equation as

$F^i(V^i, t) = 0$, where

$$F^i(w,t) \equiv rw(t) - \overline{\lambda}\left(\widetilde{V}^i(t) - c - w(t)\right)_+ + \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)}\left(\overline{V} - w(t)\right) - \dot{w}(t)$$

is a functional which may be applied to arbitrary test functions $w(t)$ to compute the remainder of the HJB equation evaluated at w.

Post-investment: The continuation value $\overline{V}$ solves a simple single-agent problem analogous to the autarky case of Section 2.2, but with a choice between prospecting and immediate investment rather than between prospecting and free-riding given that $\pi_+ > 1/R$. $\overline{V}$ may be characterized explicitly as

$$\overline{V} = \max\left\{\pi_+ R - 1, \frac{\overline{\lambda}}{\overline{\lambda} + r}(h(\pi_+)(\pi_{++}R - 1) - c)\right\}.$$

Recall from Lemma 2 that r^* is defined as the minimal discount rate at which firm i invests immediately after seeing firm $-i$ invest. Thus r^* corresponds to the smallest r such that the first argument of the max operator dominates.

Post-signal: Following observation of a high signal, firm i's continuation payoff $\widetilde{V}^i(t)$ is bounded below by the payoff of investing immediately if the signal is high, and never investing otherwise. Thus $\widetilde{V}^i(t) \geq h(\mu^i(t))(\mu^i_+(t)R - 1)$. Some algebra yields the useful associated identity

$$h(\mu)(\mu_+ R - 1) - c = K(\mu - \pi_A),$$

where $K \equiv q^H(R-1) + (1 - q^L) > 0$. Thus $\widetilde{V}^i(t) - c \geq K(\mu^i(t) - \pi_A)$, and the inequality holds with equality if firm i optimally invests immediately upon obtaining a high signal at time t.

Lemma C.1. $\overline{V} \leq K(\pi_+ - \pi_A)$.

Proof. If $r \leq r^*$, then $\overline{V} = \frac{\overline{\lambda}}{\overline{\lambda}+r}K(\pi_+ - \pi_A)$, in which case $\overline{V} < K(\pi_+ - \pi_A)$. Otherwise, $\overline{V} = \pi_+ R - 1$, and so by Assumption 3, $\overline{V} \leq K(\pi_+ - \pi_A)$. □

D Proofs

D.1 Proof of Lemma 1

The requirement that $\pi_0 > \pi_A$ for every $c \leq \overline{c}$ is equivalent to the condition $h(\pi_0)(\pi_+ R - c) > \overline{c}$. By the law of total probability,

$$\pi_+ R - 1 = h(\pi_+)(\pi_{++} R - 1) + l(\pi_+)(\pi_{+-} R - 1),$$

so that $\overline{c} = -l(\pi_+)(\pi_{+-} R - 1)$ and the desired condition may be stated as $\phi(R) > 0$, where

$$\phi(R) \equiv h(\pi_0)(\pi_+ R - 1) + l(\pi_+)(\pi_{+-} R - 1).$$

Note that $\phi(R)$ is strictly increasing in R, and $\phi(1/\pi_+) = l(\pi_+)(\pi_{+-}/\pi_+ - 1) < 0$ while $\phi(1/\pi_{+-}) = h(\pi_0)(\pi_+/\pi_{+-} - 1) > 0$. Further, for any $R < 1/\pi_{+-}$,

$$l(\pi_+)(\pi_{+-} R - 1) > l(\pi_0)(\pi_{+-} R - 1) > l(\pi_0)(\pi_- R - 1),$$

in which case $\phi(1/\pi_0) > \pi_0 R - 1$. So if $1/\pi_0 < 1/\pi_{+-}$ we have $\phi(1/\pi_0) > 0$. Thus $\phi(1/\max\{\pi_0, \pi_{+-}\}) > 0$. It follows that there exists a unique R_0, bounded between $1/\pi_+$ and $1/\max\{\pi_0, \pi_{+-}\}$, at which ϕ crosses zero, as desired.

D.2 Proof of Proposition 1

Fix a firm i, and suppose firm $-i$ follows its equilibrium strategy. We first show that firm i's equilibrium prospecting policy is a best response. By Lemma D.10, firm i's optimal policy must be a threshold rule, so it remains only to argue that $T_i^* = \infty$ is the optimal threshold. Consider any time $t > T^A$ and history in which firm i has obtained a High signal. Because $\mu^i(t) = \pi_A$, and $h(\pi_A)(\pi_{A+} R - 1) > 0$, it follows that $\mu^i_+(t) > 1/R$. So investing immediately when in possession of a High signal at any time, which yields a payoff of $\mu^i_+(t) R - 1 > 0$, dominates waiting until firm $-i$ invests, which yields a payoff of 0 (because firm $-i$ never invests). As this argument holds for arbitrary large $t > T^A$, it must be that $T_i^* = \infty$ is optimal.

It remains to verify that firm i's optimal prospecting policy prior to obtaining a signal is a threshold policy with $\overline{T}_i = T^A$. Subsequent to the cutoff time T^A the firm is in autarky with beliefs π_A, so $\lambda^i(t) = 0$ is trivially an optimal strategy from this point onward. So consider times prior to T^A. Let $V^\dagger(t) \equiv K(\mu^{\overline{\lambda}}(t) - \pi_A)$, where K is as defined in Appendix C. Inserting $V^\dagger$ into the function F^i defined in Appendix C, and using the fact that $\widetilde{V}^i(t) = V^\dagger(t) + c$ for all t given that $\mu^i = \mu^{\overline{\lambda}}$ and $T_i^* = \infty$, we have

$$F^i(V^\dagger, t) = rV^\dagger(t) + \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)}(\overline{V} - K(\pi_+ - \pi_A)).$$

Note that for $t < T^A$, $V^\dagger(t) > 0$ and $\dot{\mu}^{\overline{\lambda}}(t) < 0$. Meanwhile Lemma C.1 in Appendix C establishes the bound $\overline{V} \leq K(\pi_+ - \pi_A)$. So $F^i(V^\dagger, t) > 0$ for times $t < T^A$.

Now, note that $V^\dagger(T^A) = 0$ by definition of T^A, while also $V^i(T^A) = 0$ given that firm i is in autarky with beliefs π_A subsequent to T^A. Therefore $V^\dagger(T^A) = V^i(T^A)$. This boundary condition, combined with the fact that $F^i(V^\dagger, t) > F^i(V^i, t) = 0$ for all $t < T^A$, implies by a standard result regarding supersolutions of ODEs that $V^\dagger(t) > V^i(t)$ for all $t \in [0, T^A]$. Then as $\widetilde{V}^i(t) \geq V^\dagger(t) + c$, prospecting at the maximum rate prior to T^A is an optimal strategy.

D.3 Proof of Proposition 2

We first characterize the follower's best response to the leader. This characterization is built around a pair of belief thresholds which pin down the times at which the follower stops prospecting and investing.

Let

$$\Delta_I(\mu) \equiv \frac{\mu_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}} \frac{\overline{\lambda}}{\overline{\lambda} + r}(\pi_{++}R - 1) - (\mu_+ R - 1).$$

As will be shown later, Δ_I represents the difference in payoffs between waiting and investing immediately following receipt of a High signal when current beliefs are μ.

Lemma D.1. *Δ_I is a strictly decreasing function of μ, and $\Delta_I(\pi_-) > 0$. Also,*

$$\Delta_I(\pi_0) = \frac{\overline{\lambda}}{\overline{\lambda} + r} h(\pi_+)(\pi_{++}R - 1) - (\pi_+ R - 1).$$

In particular, $\Delta_I(\pi_0) > 0$ whenever $r \leq r^*$.

Proof. Differentiating Δ_I yields

$$\Delta_I'(\mu) = \left(\frac{1}{\pi_{++} - \pi_{+-}} \frac{\overline{\lambda}}{\overline{\lambda} + r}(\pi_{++}R - 1) - R \right) \frac{d\mu_+}{d\mu}.$$

By assumption $\pi_{+-} < 1/R < \pi_{++}$, so

$$\Delta'(\mu) < -\frac{r}{\overline{\lambda}+r} R \frac{d\mu_+}{d\mu} < 0.$$

Further, $\Delta_I(\pi_-) = -(\pi_{+-}R - 1) > 0$. Finally, to simplify $\Delta_I(\pi_0)$ use the law of total probability to write $\pi_+ = h(\pi_+)\pi_{++} + (1 - h(\pi_+))\pi_{+-}$, or equivalently $\pi_+ - \pi_{+-} = h(\pi_+)(\pi_{++} - \pi_{+-})$. This identity may be used to write $\Delta_I(\pi_0)$ in the desired form. $\square$

In light of the previous lemma, define the investment belief threshold $\mu^* \in (\pi_-, \pi_0]$ as follows:

$$\mu^* \equiv \begin{cases} \pi_0, & \Delta_I(\pi_0) \geq 0, \\ \Delta_I^{-1}(0), & \Delta_I(\pi_0) < 0. \end{cases}$$

Define the associated investment time threshold $T_F^* \equiv (\mu^{\overline{\lambda}})^{-1}(\mu^*)$. This threshold is uniquely defined given that $\mu^{\overline{\lambda}}$ is a strictly decreasing function satisfying $\mu^{\overline{\lambda}}(0) = \pi_0$ and $\mu^{\overline{\lambda}}(\infty) = \pi_-$.

Next, define

$$\Delta_P(\mu) \equiv \frac{\mu - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda}+r} \overline{V} - (\check{V}(\mu) - c),$$

where

$$\check{V}(\mu) \equiv h(\mu) \max\left\{\mu_+ R - 1, \frac{\mu_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}} \frac{\overline{\lambda}}{\overline{\lambda}+r} (\pi_{++}R - 1)\right\}.$$

We will see later that Δ_P represents the difference in payoffs between prospecting or not when current beliefs are μ, and $\check{V}(\mu)$ represents the average continuation value after obtaining a signal at beliefs μ.

We note two important properties of $\check{V}$. First, the argument of the max operator which dominates depends on the size of μ relative to μ^*, with the first argument dominating when $\mu > \mu^*$, while otherwise the second argument dominates. (When $\Delta(\pi_0) \leq 0$, the two branches are equal when $\mu = \mu^*$. Otherwise, the second argument dominates when $\mu = \mu^*$.) Second, $\check{V}(\mu)$ can be rewritten using Lemma D.2 as

$$\check{V}(\mu) \equiv \max\left\{h(\mu)(\mu_+ R - 1), \frac{\mu - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda}+r} h(\pi_+)(\pi_{++}R - 1)\right\},$$

a form which will be convenient for various proofs.

Lemma D.2. *For every* $\mu \in [\pi_-, \pi_+]$,

$$h(\pi_+)\frac{\mu - \pi_-}{\pi_+ - \pi_-} = h(\mu)\frac{\mu_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}}.$$

Proof. Note that both the lhs and rhs of the identity in the lemma statement are affine functions of μ. (The lhs is immediate, while the numerator of the rhs may be rewritten $q^H\mu - \pi_{+-}h(\mu)$, which is affine in μ given that $h(\mu)$ is.) It is therefore enough to show that they coincide at two distinct values of μ. Note that when $\mu = \pi_-$, both sides vanish, while when $\mu = \pi_+$, both sides reduce to $h(\pi_+)$, as desired. □

Lemma D.3. Δ_P *is a strictly decreasing function and* $\Delta_P(\pi_-) > 0$.

Proof. Let

$$\hat{\Delta}(\mu) \equiv \frac{\mu - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda} + r}(\overline{V} - h(\pi_+)(\pi_{++}R - 1)) + c.$$

Differentiate $\hat{\Delta}$ to obtain

$$\hat{\Delta}'(\mu) = \frac{1}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda} + r}(\overline{V} - h(\pi_+)(\pi_{++}R - 1)).$$

By Lemma C.1, $\overline{V} \le K(\pi_+ - \pi_A)$, i.e. $\overline{V} - h(\pi_+)(\pi_{++}R - 1) \le -c$, and so $\hat{\Delta}'(\mu) < 0$ for all μ.

Clearly $\Delta_P(\mu) = \hat{\Delta}(\mu)$ for $\mu \le \mu^*$. Meanwhile $\Delta_P(\mu) \le \hat{\Delta}(\mu)$ for $\mu > \mu^*$. Clearly $\Delta_P'(\mu) < 0$ for $\mu < \mu^*$. Meanwhile as Δ_P is continuous at μ^* and an affine function of μ on $[\mu^*, \pi_0]$, to ensure $\Delta_P \le \hat{\Delta}$ it must be that $\Delta_P'(\mu) = \Delta_P'(\mu^*_+) \le \hat{\Delta}'(\mu^*) < 0$ for $\mu \in (\mu^*, \pi_0]$. Hence Δ_P is a strictly decreasing function. Finally, note that $\Delta_P(\pi_-) = c > 0$. □

In light of the previous lemma, define the prospecting belief threshold $\overline{\mu} \in (\pi_-, \pi_0]$ by

$$\overline{\mu} \equiv \begin{cases} \pi_0, & \Delta_P(\pi_0) \ge 0, \\ \Delta_P^{-1}(0), & \Delta_P(\pi_0) < 0. \end{cases}$$

Define the associated prospecting time threshold $\overline{T}_F \equiv (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$.

We now show that the follower's strategy is a best response to the leader's strategy. We further show that the best response is unique, which will be important for proving Proposition 3.

Lemma D.4. *Suppose firm $-i$ chooses the threshold strategy $T^*_{-i} = \overline{T}_{-i} = \infty$. Then firm i's unique best response is the threshold strategy characterized by $T^*_i = T^*_F$ and $\overline{T}_i = \overline{T}_F$.*

Proof. Given firm $-i$'s strategy, firm i's posterior beliefs satisfy $\mu^i(t) = \mu^{\overline{\lambda}}(t)$ for all time. Consider first firm i's optimal investment policy. Lemma D.10 establishes that an optimal policy must be a threshold rule, and so at each point in time either $V^i_+(t) = W^\dagger(t) \equiv \mu^{\overline{\lambda}}_+(t)R - 1$, or else $V^i_+(t) = W^\ddagger(t)$, where $W^\ddagger(t)$ is the value of investing immediately after the leader invests. The follower's investment cutoff time is determined by the first time at which $W^\dagger(t)$ falls below $W^\ddagger(t)$.

The value $W^\ddagger(t)$ may be calculated explicitly as

$$W^\ddagger(t) = \nu^{\overline{\lambda}}_+(t)\frac{\overline{\lambda}}{\overline{\lambda}+r}h(\pi_+)(\pi_{++}R - 1),$$

where $\nu^{\overline{\lambda}}_+(t)$ is firm i's posterior belief that firm $-i$ has not yet received a signal, given that $-i$ has not yet invested and firm i's signal is High, and the remainder of the expression is the expected discounted value of waiting for firm $-i$ to acquire a signal and invest. This argument additionally establishes that $\widetilde{V}^i(t) = h(\mu^{\overline{\lambda}}(t))V^i_+(t) = \check{V}(\mu^{\overline{\lambda}}(t))$ for all t.

Variants of Lemmas B.1 and B.2 applied to an environment with initial beliefs π_+ can be used to obtain two expressions for $\dot{\mu}^{\overline{\lambda}}(t)$; equating these two expressions yields $\nu^{\overline{\lambda}}_+(t) = (\mu^{\overline{\lambda}}_+(t) - \pi_{+-})/(h(\pi_+)(\pi_{++} - \pi_{+-}))$. Comparing $W^\ddagger(t) - W^\dagger(t)$ with $\Delta_I(\mu)$, we see that $W^\dagger(t)$ falls below $W^\ddagger(t)$ at the time t such that $\mu^{\overline{\lambda}}(t) = \mu^*$. So $T^*_i = T^*_F$ is firm i's unique optimal investment threshold.

We now derive firm i's optimal prospecting strategy. Recall that in Appendix C we showed that the HJB equation satisfied by firm i's continuation value function V^i may be expressed as $F^i(V^i, t) = 0$. We first consider times $t \geq \overline{T}$. Let $V^\dagger(t) \equiv \frac{\mu^{\overline{\lambda}}(t)-\pi_-}{\pi_+-\pi_-}\frac{\overline{\lambda}}{\overline{\lambda}+r}\overline{V}$. For all times $t \geq \overline{T}_F$, $\mu^{\overline{\lambda}}(t) \leq \overline{\mu}$ and so $\Delta_P(\mu^{\overline{\lambda}}(t)) = V^\dagger(t) - \check{V}(\mu^{\overline{\lambda}}(t)) + c \geq 0$. Since $\widetilde{V}^i(t) = \check{V}(\mu^{\overline{\lambda}}(t))$ for all time, we therefore have $V^\dagger(t) \geq \widetilde{V}^i(t) - c$ for all $t \geq \overline{T}_F$. Inserting $V^\dagger$ into F^i therefore yields

$$F^i(V^\dagger, t) = \left(\frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}\overline{\lambda} + \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)}\right)\frac{r}{\overline{\lambda}+r}\overline{V}$$

for $t \geq \overline{T}_F$. Using Lemma B.2 to eliminate $\dot{\mu}^{\overline{\lambda}}(t)$ yields $F^i(V^\dagger, t) = 0$. As $V^\dagger$ is a bounded, absolutely continuous function, it follows by a standard verification argument that $V^i(t) = V^\dagger(t)$ for $t \geq \overline{T}_F$. It follows immediately that for times $t > \overline{T}_F$, we have $\Delta_P(\mu^{\overline{\lambda}}(t)) = V^\dagger(t) - \check{V}(\mu^{\overline{\lambda}}(t)) + c = V^i(t) - \widetilde{V}^i(t) + c > 0$, meaning that $\lambda^i(t) = 0$ is firm i's unique optimal prospecting policy for times $t > \overline{T}_F$.

Now consider times $t < \overline{T}_F$. If $\overline{\mu} = \pi_0$ then this time interval is empty, so assume $\overline{\mu} < \pi_0$. Let $V^\ddagger(t) \equiv \check{V}(\mu^{\overline{\lambda}}(t)) - c$. We will show that $F^i(V^\ddagger, t) > 0$ for all $t < \overline{T}_F$, where F^i is as defined in Appendix C.

Note first that $\widetilde{V}^i(t) = \check{V}(\mu^{\overline{\lambda}}(t))$ implies that $V^\ddagger(t) = \widetilde{V}^i(t) - c$, an identity which will prove useful for evaluating $F^i(V^\ddagger, t)$. Also, $\Delta_P(\mu^{\overline{\lambda}}(\overline{T}_F)) = 0$ implies that $V^\ddagger(\overline{T}_F) = V^\dagger(\overline{T}_F)$. In particular, since $V^\dagger(\overline{T}_F) > 0$ and $V^\ddagger(t)$ is strictly decreasing in t, we have $V^\ddagger(t) > 0$ for $t < \overline{T}_F$.

Suppose $t \leq \widehat{T}_F \equiv \min\{\overline{T}_F, T_F^*\}$. On this time range $\check{V}(\mu^{\overline{\lambda}}(t)) - c = K(\mu^{\overline{\lambda}}(t) - \pi_A)$, and so $F^i(V^\ddagger, t)$ evaluates to

$$F^i(V^\ddagger, t) = rV^\ddagger(t) + \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)}(\overline{V} - K(\pi_+ - \pi_A)).$$

By Lemma C.1, $\overline{V} \leq K(\pi_+ - \pi_A)$. Further, we established above that $V^\ddagger(t) > 0$. Thus $F^i(V^\ddagger, t) > 0$.

If $\overline{\mu} \geq \mu^*$ then there are no further times to check, so suppose instead that $\overline{\mu} < \mu^*$ and $t \in (T_F^*, \overline{T}_F)$. For such times,

$$\check{V}(\mu^{\overline{\lambda}}(t)) = \frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda} + r}h(\pi_+)(\pi_{++}R - 1) = \frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda} + r}(K(\pi_+ - \pi_A) + c).$$

This expression, combined with the identity derived in Lemma B.2, allows us to evaluate $F^i(V^\ddagger, t)$ as

$$F^i(V^\ddagger, t) = -\overline{\lambda}\frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}(\overline{V} - K(\pi_+ - \pi_A)) - rc.$$

Now, $\mu^{\overline{\lambda}}(t) \in (\overline{\mu}, \mu^*)$ for $t \in (T_F^*, \overline{T}_F)$, and therefore

$$\Delta_P(\mu^{\overline{\lambda}}(t)) = \frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda} + r}(\overline{V} - K(\pi_+ - \pi_A) - c) + c < 0,$$

or equivalently

$$-\overline{\lambda}\frac{\mu^{\overline{\lambda}}(t)-\pi_-}{\pi_+-\pi_-}(\overline{V}-K(\pi_+-\pi_A))\geq(\overline{\lambda}+r)c-\frac{\mu^{\overline{\lambda}}(t)-\pi_-}{\pi_+-\pi_-}\overline{\lambda}c>rc.$$

So $F^i(V^\ddagger,t)>0$ for $t\in(T_F^*,\overline{T}_F)$.

We have established that $F^i(V^\ddagger,t)>0$ for all $T<\overline{T}_F$, and further that $V^\ddagger(\overline{T}_F)=V^\dagger(\overline{T}_F)$. Recall that we earlier established $V^\ddagger(t)=V^i(t)$ for all times $t\geq\overline{T}_F$, so $V^\ddagger(\overline{T}_F)=V^i(\overline{T}_F)$. Then as $F^i(V^i,t)=0$ for all $t\leq\overline{T}_F$, a standard result regarding supersolutions of ODEs implies that $V^\ddagger(t)>V^i(t)$ for all $t<\overline{T}$. Then the identity $V^\ddagger(t)=\widetilde{V}^i(t)-c$ implies that $\lambda^i(t)=\overline{\lambda}$ is uniquely optimal for $t\leq\overline{T}_F$. □

We now establish that the leader's strategy is a best reply to the follower's.

Lemma D.5. *Suppose that firm $-i$ employs a threshold strategy satisfying $\mu^{\overline{\lambda}}(\widehat{T}_{-i})>\pi^A$, where $\widehat{T}_i\equiv\min\{T^*_{-i},\overline{T}_{-i}\}$. Then firm i's unique best reply is the threshold strategy $T_i^*=\overline{T}_i=\infty$.*

Proof. Subsequent to time $\widehat{T}_{-i}$, firm i is in autarky with beliefs $\mu^i(t)=\mu^{\overline{\lambda}}(\widehat{T}_{-i})>\pi^A$. Thus its unique optimal policy for times $t\geq\widehat{T}_{-i}$ is to prospect at rate $\overline{\lambda}$ and invest immediately. By Lemma D.10, it follows that firm i's unique optimal investment strategy is the cutoff rule $T_i^*=\infty$. It remains only to characterize i's optimal prospecting behavior prior to time $\widehat{T}_{-i}$. Note that for such times, $\mu^i(t)=\mu^{\overline{\lambda}}(t)$.

Define $V^\dagger(t)\equiv K(\mu^{\overline{\lambda}}(t)-\pi_A)$. Since $T_i^*=\infty$, it must be that $\widetilde{V}^i(t)-c=V^\dagger(t)$ for all times. Then inserting $V^\dagger$ into the functional F^i defined in Appendix C yields

$$F^i(V^\dagger,t)=rV^\dagger(t)+\frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+-\mu^{\overline{\lambda}}(t)}(\overline{V}-K(\pi_+-\pi_A)).$$

Note that $\overline{V}\leq K(\pi_+-\pi_A)$ by Lemma C.1, so the second term on the rhs is nonnegative. Meanwhile $\mu^{\overline{\lambda}}(t)>\pi_A$ for $t\leq\widehat{T}_{-i}$, meaning $V^\dagger(t)>0$. So $F^i(V^\dagger,t)>0$ for all such $t\leq\widehat{T}_{-i}$.

Now note that as firm i is in autarky at time $\widehat{T}_{-i}$, its value function at this point is $V^i(\widehat{T}_{-i})=\frac{\overline{\lambda}}{\overline{\lambda}+r}V^\dagger(\widehat{T}_{-i})<V^\dagger(\widehat{T}_{-i})$. This boundary condition, combined with the fact that $F^i(V^\dagger,t)>0$ while $F^i(V^i,t)=0$ for all $t<\widehat{T}_i$, implies by a standard result regarding supersolutions of ODEs that $V^\dagger(t)>V^i(t)$ for all $t\in[0,\widehat{T}_i]$. Then the fact that $\widetilde{V}^i(t)-c=V^\dagger(t)$ implies that $\lambda^{-i}(t)=\overline{\lambda}$ is firm $-i$'s unique optimal prospecting policy for times $t<\widehat{T}_i$. □

Combining this result with the following lemma establishes that the leader's strategy is a best response to the follower's and completes the proof.

Lemma D.6. $\max\{\overline{\mu}, \mu^*\} > \pi_A$.

Proof. If $\overline{\mu} = \pi_0$ then the result is automatic. So assume $\overline{\mu} < \pi_0$, in which case $\overline{\mu}$ is pinned down by the condition $\Delta_P(\overline{\mu}) = 0$. If $\overline{\mu} \geq \mu^*$, then $\Delta_P(\overline{\mu}) = 0$ may be written

$$\frac{\overline{\mu} - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} \overline{V} - K(\overline{\mu} - \pi_A) = 0.$$

As $\overline{\mu} > \pi_-$, it must be that $\overline{\mu} > \pi_A$ for this equality to hold. If instead $\mu^* > \overline{\mu}$, then $\Delta_P(\mu^*) < 0$, which is equivalently

$$\frac{\mu^* - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} \overline{V} - K(\mu^* - \pi_A) < 0.$$

As $\mu^* > \pi_-$, it must be that $\mu^* > \pi_A$ for this equality to hold. $\square$

D.4 Proof of Lemma 3

Note that Δ_P, $\overline{\mu}$, T_F^* π_A, T^A, and $\overline{V}$ are each functions of c, while Δ_I, μ^*, and $\overline{T}_F$ are independent of c. Wherever a parameter depends on c, we will make that dependence explicit throughout this proof.

We begin with a series of auxiliary lemmas.

Lemma D.7. *$\overline{\mu}(c)$ is increasing in c, strictly so whenever $\overline{\mu}(c) < \pi_0$.*

Proof. Recall that

$$\Delta_P(\mu, c) = \frac{\mu - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} \overline{V} - \check{V}(\mu) + c,$$

where $\check{V}(\mu)$ is not a function of c. When $r > r^*$, $\overline{V}$ is independent of c, and so $\partial \Delta_P / \partial c = 1 > 0$. Otherwise, $\partial \overline{V} / \partial c = -\frac{\overline{\lambda}}{\overline{\lambda} + r}$, so that

$$\frac{\partial \Delta_P}{\partial c} = -\frac{\mu - \pi_-}{\pi_+ - \pi_-} \left(\frac{\overline{\lambda}}{\overline{\lambda} + r} \right)^2 + 1 > 0.$$

In either case, Δ_P is strictly increasing in c for every μ. Since Δ_P is strictly decreasing in μ, the lemma statement follows. $\square$

Lemma D.8. $\overline{\mu}(c) < \min\{\mu^*, \pi_A(c)\}$ *when* c *is sufficiently small.*

Proof. Note that $\Delta_P(\mu^*, c)$ may be written

$$\Delta_P(\mu^*, c) = \frac{\mu^* - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} \left(\overline{V}(c) - h(\pi_+)(\pi_{++}R - 1)\right) + c.$$

For all c we have $\overline{V}(c) \leq \max\left\{\pi_+ R - 1, \frac{\overline{\lambda}}{\overline{\lambda}+r} h(\pi_+)(\pi_{++}R - 1)\right\} < h(\pi_+)(\pi_{++}R - 1)$. Therefore the first term in the previous expression for $\Delta_P(\mu^*, c)$ is bounded below zero for all c, while the second term vanishes as $c \to 0$. It follows that $\Delta_P(\mu^*, c) < 0$ for sufficiently small c, in which case $\overline{\mu}(c) < \mu^*$.

Next, note that when $c = 0$, $\pi_A(c)$ satisfies $h(\pi_A(c))(\pi_{A+}(c)R - 1) = 0$, i.e. $\pi_{A+}(c) = 1/R$. Hence $\pi_{A+}(c) > \pi_-$ given that $\pi_{+-} < 1/R$. Additionally, using the fact that $\check{V}(\mu) \geq \frac{\mu - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda}+r} h(\pi_+)(\pi_{++}R - 1)$ for all μ, we must have

$$\Delta_P(\pi_A(c), c) \leq \frac{\pi_A(c) - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} (\overline{V}(c) - h(\pi_+)(\pi_{++}R - 1)) + c.$$

Given that $\pi_A(0) > \pi_-$, an argument very similar to the one in the previous paragraph shows that this upper bound is negative for sufficiently small c, implying that $\overline{\mu}(c) < \pi_A(c)$ for such costs. □

Lemma D.9. *If* r *is sufficiently small, then* $\overline{\mu}(c) = \pi_0$ *for costs sufficiently close to* $\overline{c}$.

Proof. Note that for sufficiently small r, $\Delta_I(\pi_0) > 0$ and $\mu^* = \pi_0$. For such a choice of r, we have

$$\Delta_P(\pi_0, c) = \frac{\pi_0 - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} \left(\overline{V}(c) - h(\pi_+)(\pi_{++}R - 1)\right) + c.$$

Since $\overline{V}(c) \geq \pi_+ R - 1$, this expression is bounded below as

$$\Delta_P(\pi_0, c) \geq c - \frac{\pi_0 - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} \overline{c}.$$

When $c = \overline{c}$, this bound is strictly positive, implying $\overline{\mu}(c) = \pi_0$, as desired. □

We now establish existence of a cost threshold c^* with the stated properties. Lemma D.7 implies that $\overline{T}_F(c)$ is decreasing in c, and is strictly decreasing whenever it is strictly positive. Further, Lemma D.8 implies that $\overline{T}_F(c) > T_F^*$ when c is sufficiently small. Then either there exists a $c^* \in (0, \bar{c})$ below which $\overline{T}_F(c) > T_F^*$ and above which $\overline{T}_F(c) \leq T_F^*$ (with the inequality possibly weak if $T_F^* = 0$) or else $\overline{T}_F(c) > T_F^*$ for every $c \leq \bar{c}$. Letting $c^* = \bar{c}$ in the latter case, this choice of c^* satisfies the properties claimed in the lemma. Further, Lemma D.9 establishes that when r is sufficiently small, $\overline{T}_F(c) = 0$ for c sufficiently close to $\bar{c}$. But then for such c it is automatically the case that $\overline{T}_F(c) \leq T_F^*$, meaning that $c^* < \bar{c}$.

We next establish existence of a cost threshold c_* with the stated properties. For this result, it is sufficient to show that the function $\Delta_P(\pi_A(c), c)$ is negative for sufficiently small c, positive for $c \in [c^*, \bar{c}]$, and crosses zero exactly once on the interval $(0, c^*)$, for then the crossing point c_* will satisfy satisfy the desired properties.

Lemma D.8 implies that $\Delta_P(\pi_A(c), c) < 0$ for c sufficiently small. And if $c^* < \bar{c}$, then when $c \geq c^*$ we have $\overline{\mu}(c) \geq \mu^*$. But by Lemma D.6, $\max\{\overline{\mu}, \mu^*\} > \pi_A(c)$, so we must have $\overline{\mu}(c) > \pi_A(c)$, i.e. $\Delta_P(\pi_A(c), c) > 0$. On the other hand if $c^* = \bar{c}$, then $\overline{\mu}_F(\bar{c}) \leq \mu^*$ and Lemma D.6 requires that $\mu^* > \pi_A(\bar{c})$, in which case

$$\Delta_P(\pi_A(\bar{c}), \bar{c}) = \frac{\pi_A(\bar{c}) - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} (\overline{V}(c) - h(\pi_+)(\pi_{++}R - 1)) + \bar{c}.$$

Using the lower bound $\overline{V}(c) \geq \pi_+ R - 1$, we can bound $\Delta_P(\pi_A(\bar{c}), \bar{c})$ below as

$$\Delta_P(\pi_A(\bar{c}), \bar{c}) \geq \bar{c} \left(1 - \frac{\pi_A(\bar{c}) - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} \right) > 0,$$

ensuring that $\Delta_P(\pi_A(c^*), c^*) > 0$ in all cases.

It remains only to show that $\Delta_P(\pi_A(c), c)$ satisfies single-crossing on $(0, c^*)$. On this interval we have $\overline{\mu}(c) < \mu^*$ by definition of c^*, and so also $\mu^* > \pi_A(c)$ by Lemma D.6. Thus

$$\Delta_P(\pi_A(c), c) = c - \frac{\pi_A(c) - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} (h(\pi_+)(\pi_{++}R - 1) - \overline{V}(c))$$

for $c \in (0, c^*)$. Define the threshold $\underline{c} \geq 0$ to be the smallest c for which $\overline{V}(c) = \pi_+ R - 1$, in case this cost threshold is positive; and otherwise set $\underline{c} = 0$. Then for

$c \in (0, c^*)$ we may write

$$\Delta_P(\pi_A(c), c) = \begin{cases} \underline{\chi}(c), & c < \underline{c} \\ \overline{\chi}(c), & c \geq \underline{c} \end{cases}$$

where

$$\underline{\chi}(c) \equiv c - \frac{\pi_A(c) - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} \left(h(\pi_+)(\pi_{++}R - 1) - \frac{\overline{\lambda}}{\overline{\lambda} + r} \left(h(\pi_+)(\pi_{++}R - 1) - c \right) \right)$$

and

$$\overline{\chi}(c) \equiv c - \frac{\pi_A(c) - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} \overline{c}.$$

As $\Delta_P(\pi_A(c), c)$ is continuous across the interface $c = \underline{c}$, it is sufficient to establish that each of $\underline{\chi}(c)$ and $\overline{\chi}(c)$ crosses zero at most once on $(0, c^*)$, and that any such crossing is from below.

Recall that $\pi_A(c)$ is the solution to $h(\mu)(\mu_+ R - 1) = c$. This is a linear equation for μ, and its solution is affine in c. Hence $\overline{\chi}(c)$ is also affine in c. Further, $\overline{\chi}(0) < 0$ while $\overline{\chi}(\overline{c}) > 0$. So $\overline{\chi}(c)$ is a strictly increasing affine function on $[0, \overline{c}]$, ensuring that it crosses zero at most once on $(0, c^*)$, from below.

Meanwhile $\underline{\chi}(c)$ is a concave quadratic in c which satisfies $\underline{\chi}(0) < 0$. If we can find some $c^\dagger > c^*$ such that $\underline{\chi}(c^\dagger) > 0$, then $\underline{\chi}(c)$ is assured to cross 0 at most once on $(0, c^*)$, from below. Choosing $c^\dagger = h(\pi_+)(\pi_{++}R - 1)$, which satisfies $c^\dagger > \overline{c} \geq c^*$, we have

$$\underline{\chi}(c^\dagger) = \left(1 - \frac{\pi_A(c^\dagger) - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} \right) h(\pi_+)(\pi_{++}R - 1).$$

Now, by definition of $\pi_A(c)$ we have $\pi_A(c^\dagger) = \pi_+$. Therefore $\underline{\chi}(c^\dagger) > 0$, as desired.

D.5 Proof of Proposition 3

Throughout this proof, fix an equilibrium strategy profile. For each firm i, define the time thresholds $t_i^A \equiv \inf\{t \ : \ \mu^i(t) \leq \pi_A\}$, $t_i^{+,0} \equiv \inf\{t \ : \ \mu_+^i(t) \leq 1/R\}$, and $t_i^{+,00} \equiv \inf\{t \ : \ \mu_+^i(t) < 1/R\}$. In other words, t_i^A is the first time firm i's beliefs reach the autarky threshold; $t_i^{+,0}$ is the first time its beliefs reach the threshold $1/R$; and $t_i^{+,00}$ is the first time its beliefs fall below $1/R$. Note that in general $t_i^{+,00} \geq t_i^{+,0}$, and the inequality is strict iff i's beliefs remain constant at $1/R$ over some time interval.

We begin by establishing that, up to a technicality, each firm must use a threshold rule for investment.

Lemma D.10. *Suppose that firm i has obtained a High signal.*

- *If $t_i^{+,00} < \infty$, there exists a cutoff time $T_i^* \leq t_i^{+,0}$ such that firm i invests immediately if $t < T_i^*$, and invests only after seeing firm $-i$ invest if $t > T_i^*$.*
- *If $t_i^{+,00} = \infty$, firm i invests immediately if $t < t_i^{+,0}$.*

Proof. Fix any firm i and time $t \leq t_i^{+,0}$. Suppose that given firm $-i$'s strategy, there exists a best reply for firm i which involves investing immediately at time t, supposing the firm has not invested yet or observed the other firm invest. Then there must be a best reply which, beginning at any time $t' < t$, involves waiting no longer than time t to invest. But the payoff of such a strategy is just a discounted version of the payoff of investing at time t', as the investment happens regardless of any information gained from firm $-i$ between times t' and t. Since $t' < t_i^{+,0}$, this payoff is strictly positive, and so it must be suboptimal to delay beyond time t'. So firm i's strategy must involve immediate investing at every time $t' < t$.

Meanwhile, it is trivially suboptimal for the firm to invest at any time $t > t_i^{+,00}$ prior to observing investment. If additionally $t_i^{+,00} < \infty$, it must also be subptimal for the firm to invest at any time $t \in [t_i^{+,0}, t_i^{+,00}]$, since at breakeven beliefs the firm makes no profits from investing immediately, but makes strictly positive profits with positive probability by waiting to see if firm $-i$ invests.

Therefore if $t_i^{+,00} < \infty$, letting T_i^* be the supremum of times at which investing immediately is a best reply for firm i, it must be that $T_i^* \leq t_i^{+,0}$, and firm i must invest immediately at all times prior to T_i^*, while it must never invest prior to seeing investment subsequent to time T_i^*.

On the other hand, if $t_i^{+,00} = \infty$, it must be a best reply for firm i to invest immediately at any time $t \geq t_i^{+,0}$, since beliefs remain at the breakeven level forever subsequent to this time. Thus if $t_i^{+,0} < \infty$, then firm i's strategy must involve immediate investing at every time $t < t_i^{+,0}$. And if $t_i^{+,0} = \infty$, then investment is strictly profitable at all times no matter what information arrives, and so any delay is suboptimal. Thus again firm i's strategy must involve immediate investing prior to $t_i^{+,0} = \infty$. □

This lemma ensures that firms' investment policies must take the form of threshold rules, except in the case that $t_i^0 < t_i^{00} = \infty$. However, multiplicity of best replies in that case impacts outcomes only off-path, as on-path the firm either obtained a High signal prior to t_i^0 and invested immediately; or else it obtained no signal prior to t_i^0, after which a signal is valueless and the firm does not optimally acquire one. Therefore any choice of a non-threshold investment policy in this case has no impact on equilibrium outcomes.

Our proof will proceed by restricting attention to equilibria in threshold investment strategies, with each firm's investment threshold denoted by T_i^*. This analysis will characterize all possible equilibrium paths, and in particular will establish that either $t_i^{00} < \infty$ or else $t_i^0 = \infty$ in any equilibrium, proving that all equilibria involve threshold investment policies.

Now, assume that both players use pure prospecting strategies. We will maintain this assumption until the end of the proof, when we verify that no equilibria with mixed prospecting strategies can exist.

We next establish an important technical result about the dynamics of the value of effort prior to time t_i^A. This result will be critical to establishing that firms follow a threshold prospecting rule in any equilibrium. For each firm i, define $f_i(t) \equiv V^i(t) - K(\mu^i(t) - \pi_A)$. Note that $f_i(t) \geq V^i(t) - \widetilde{V}^i(t) + c$, with equality for all $t < T_i^*$.

Lemma D.11. *Fix any firm i. Then for almost every $t \in [0, \min\{T_i^*, t_i^A\}]$, either $f_i(t) < 0$ or $f_i'(t) > 0$.*

Proof. Fix a firm i. Suppose first that $T_{-i}^* \leq t < t_i^A$. Then at time t firm i is in autarky with beliefs $\mu^i(t) > \pi_A$, meaning its continuation value is $V^i(t) = \frac{\overline{\lambda}}{\overline{\lambda}+r} K(\mu^i(t) - \pi_A) < K(\mu^i(t) - \pi_A)$. Thus $f_i(t) < 0$ for all such times. So it is sufficient to establish the result for $t < \min\{t_i^A, T_i^*, T_{-i}^*\}$.

Note that whenever $t < T_i^*$, we have $f_i(t) = V^i(t) - \widetilde{V}^i(t) + c$. Then for almost every $t < \min\{T_i^*, T_{-i}^*\}$ such that $f_i(t) \geq 0$, $V^i(t)$ must satisfy the HJB equation

$$rV^i(t) = \lambda^{-i}(t) \frac{\mu^i(t) - \pi_-}{\pi_+ - \pi_-} (\overline{V} - V^i(t)) + \dot{V}^i(t).$$

This may be rewritten in terms of f and f' as

$$f_i'(t) = rK(\mu^i(t) - \pi_A) - \lambda^{-i}(t)\frac{\mu^i(t) - \pi_-}{\pi_+ - \pi_-}(\overline{V} - K(\pi_+ - \pi_A) - f_i(t)) + rf_i(t).$$

The first term on the rhs of this expression is strictly positive for every $t < t_i^A$. Further, by Lemma C.1, $\overline{V} \leq K(\pi_+ - \pi_A)$. Finally, the coefficient on $f_i(t)$ on the rhs is always non-negative. Thus whenever $f_i(t) \geq 0$, we must have $f_i'(t) > 0$. □

We proceed by splitting the analysis into two cases: either $T_i^* < \infty$ for some firm i, or else $T_1^* = T_2^* = \infty$. We will show that in the first case, the only permissible equilibrium behavior is the leader-follower strategy profile, while in the second case, the only permissible behavior is the symmetric equilibrium profile. Consider first the $T_i^* < \infty$ case. The following lemma establishes that the remaining firm $-i$ must employ the leader strategy in any equilibrium.

Lemma D.12. *Suppose that $T_i^* < \infty$ for some firm i. Then firm $-i$ must follow the threshold strategy $\overline{T}_{-i} = T_{-i}^* = \infty$.*

To establish this result, we first prove an auxiliary lemma which restricts the permissible scope of equilibrium behavior and beliefs in response to a firm using a threshold investment rule with $T_i^* < \infty$.

Lemma D.13. *Suppose that $T_i^* < \infty$ for some firm i. Then $T_{-i}^* = \infty$ and $\mu^{-i}(T_i^*) > \pi_A$.*

Proof. Suppose by way of contradiction that $\mu^{-i}(T_i^*) < \pi_A$. Then beginning at time T_i^*, firm $-i$ is in autarky with beliefs below the autarky threshold, implying that it does not invest on the equilibrium path after time T_i^*. Further, on the interval $(t_{-i}^A, T_i^*]$, we have $V^{-i}(t) \geq 0 > K(\mu^i(t) - \pi_A)$. Then at all such times, it cannot be optimal for firm i to both prospect and invest immediately upon acquiring a signal. Therefore firm $-i$ is in autarky beginning at time t_{-i}^A.

But since $t_{-i}^A < T_i^*$ by continuity of μ^{-i}, the fact that it is optimal for firm i to invest immediately at times in $[t_{-i}^A, T_i^*)$ but wait after T_i^* implies that $\mu_+^i(t_{-i}^A) = \mu_+^i(T_i^*) = 1/R$. Therefore $\mu^i(t_{-i}^A) < \pi_A$, so $\lambda^i(t) = 0$ for all $t \geq t_{-i}^A$. But then on the equilibrium path firm i does not invest first after t_{-i}^A, implying firm $-i$ is in autarky with constant beliefs $\mu^{-i}(t) = \mu^{-i}(t_{-i}^A) = \pi_A$ for all times $t > t_{-i}^A$. This contradicts $\mu^{-i}(T_i^*) < \pi_A$, so it must be that $\mu^{-i}(T_i^*) \geq \pi_A$, and in particular $T_i^* \geq t_i^A$.

Subsequent to time T_i^*, firm $-i$ is in autarky with fixed beliefs no lower than the autarky threshold. Therefore $\mu_+^{-i}(t) > 1/R$ for all $t \geq T_i^*$, in which case immediate investing is strictly superior to waiting forever for every $t \geq T_i^*$. Thus firm $-i$ must choose $T_{-i}^* = \infty$.

Now suppose by way of contradiction that $\mu^{-i}(T_i^*) = \pi_A$, in which case $t_{-i}^A \leq T_i^*$. As firm $-i$'s beliefs do not change over the interval $[t_{-i}^A, T_i^*]$, it must be in autarky with constant beliefs π_A from time t_{-i}^A onward, implying $V^{-i}(t) = 0$.

Note that $V^{-i}(t_{-i}^A) = 0$ and $\mu^{-i}(t_{-i}^A) = \pi_A$ imply $f_{-i}(t_{-i}^A) = 0$. But by Lemma D.11, for almost every $t \in [0, t_{-i}^A]$ either $f_{-i}(t) < 0$ or $f'_{-i}(t) > 0$. These conditions imply that if $f_{-i}(t) = 0$ for some $t < t_{-i}^A$, then $f_{-i}(t) > 0$ for all $t' \in (t, t_{-i}^A]$. Hence $f_{-i}(t) < 0$ for all $t < t_{-i}^A$, implying that for all such times the value of waiting is less than the value of prospecting and investing immediately upon obtaining a High signal. Since this investment strategy is a lower bound on the value of prospecting, it must be that $\lambda^{-i}(t) = \overline{\lambda}$ a.e. on $[0, t_{-i}^A]$.

This prospecting policy, combined with $T_{-i}^* = \infty$, implies that $\mu^i(t) \leq \mu^{-i}(t)$ for $t \in [0, t_{-i}^A]$ and therefore $t_i^A \leq t_{-i}^A$. If $t_i^A < t_{-i}^A$, then for every $t \in (t_i^A, t_{-i}^A)$, firm $-i$'s prospecting and investment policies imply that $\mu^i(t) < \pi_A$, meaning it cannot be optimal for firm i to both prospect and invest immediately upon obtaining a signal at any such time. Thus firm i does not invest on the equilibrium path on this time interval, implying μ^{-i} is constant on the interval, contradicting the definition of t_{-i}^A. So $t_1^A = t_2^A = t^A$ for some t^A, which can only hold if $T_i^* \geq t^A$ and $\lambda^i(t) = \overline{\lambda}$ for almost every $t \in [0, t^A]$.

If $V^i(t^A) > 0$, then given continuity of V^i and μ^i, for sufficiently large $t < t^A$ it would be the case that $V^i(t) > K(\mu^i(t) - \pi_A)$. But then it cannot be optimal for firm i to both prospect and invest immediately at such times, a contradiction. So $V^i(t^A) = 0$. But as $T_{-i}^* = \infty$, this can be true only if $\lambda^{-i}(t) = 0$ for a.e. $t > t^A$. But then subsequent to time T^A, firm i is in autarky with beliefs $\mu^i(t) = \pi_A$, contradicting the optimality $T_1^* < \infty$. So $\mu^{-i}(T_i^*) > \pi_A$, as desired. □

Proof of Lemma D.12. Lemma D.13 establishes that $T_{-i}^* = \infty$ and $\mu^{-i}(T_i^*) > \pi^A$. The latter inequality implies that for $t > T_i^*$, firm $-i$ is in autarky with beliefs above the autarky threshold, meaning $-i$'s unique optimal prospecting policy subsequent to T_i^* is $\lambda^{-i}(t) = \overline{\lambda}$. It remains only to pin down firm $-i$'s optimal prospecting behavior prior to T_i^*.

Define $V_{-i}^{\dagger}(t) \equiv K(\mu^i(t) - \pi_A)$. Since $T_{-i}^* = \infty$, it must be that $\widetilde{V}^{-i}(t) - c = V_{-i}^{\dagger}(t)$

for all times. Then inserting $V^\dagger_{-i}$ into the functional F^{-i} defined in Appendix C yields

$$F^{-i}(V^\dagger_{-i}, t) = rV^\dagger_{-i}(t) + \frac{\dot{\mu}^{-i}(t)}{\pi_+ - \mu^{-i}(t)}(\overline{V} - K(\pi_+ - \pi_A)).$$

Note that $\overline{V} \leq K(\pi_+ - \pi_A)$ by Lemma C.1, so the second term on the rhs is non-negative. Meanwhile for $t \leq T_i^*$, $\mu^{-i}(t) > \pi_A$ and therefore $V^\dagger_{-i}(t) > 0$. Thus $F^{-i}(V^\dagger_{-i}, t) > 0$ for all times $t \leq T_i^*$.

Now note that as firm $-i$ is in autarky at time T_i^*, its value function at this point is $V^{-i}(T_i^*) = \frac{\overline{\lambda}}{\overline{\lambda}+r} V^\dagger_{-i}(T_i^*) < V^\dagger_{-i}(T_i^*)$. This boundary condition, combined with the fact that $F^{-i}(V^\dagger_{-i}, t) > 0$ while $F^{-i}(V^{-i}, t) = 0$ all $t \in [0, T_i^*]$, implies by a standard result regarding supersolutions of ODEs that $V^\dagger_{-i}(t) > V^{-i}(t)$ for all $t \leq T_i^*$. Then as $V^\dagger_{-i}(t) = \widetilde{V}^{-i}(t) - c$, $\lambda^{-i}(t) = \overline{\lambda}$ is firm $-i$'s unique optimal prospecting strategy prior to T_i^*. □

Lemma D.12 establishes that in any equilibrium in threshold investment strategies in which some $T_i^* < \infty$, the other firm must follow the leader's strategy. Meanwhile Lemma D.4 establishes that the follower's strategy is a unique best reply to the leader's strategy. So there exists a unique equilibrium in threshold investment strategies with some $T_i^* < \infty$, namely the leader-follower equilibrium.

The following lemma treats the remaining case, in which $T_1^* = T_2^* = \infty$. It establishes that the symmetric equilibrium strategies are the only ones consistent with equilibrium in this case.

Lemma D.14. *Suppose $T_1^* = T_2^* = \infty$. Then both firms follow threshold prospecting policies with $\overline{T}_1 = \overline{T}_2 = T^A$.*

Proof. Note that when $T_1^* = T_2^* = \infty$, we have $f_i(t) = V^i(t) - \widetilde{V}^i(t) + c$ for every i and t, and so a firm's optimal prospecting rate depends only on the sign of $f_i(t)$. Further, $f_i(t) \geq 0$ whenever $t \geq t_i^A$, and the inequality is strict if either $\mu^i(t) < \pi_A$ or $V^i(t) > 0$. Also, by Lemma D.11, for each firm i and almost every $t < t_i^A$ either $f_i(t) < 0$ or $f_i'(t) > 0$.

Suppose first that $t_i^A < t_{-i}^A$ for some firm i. Define $t_i^{AA} \equiv \inf\{t \,:\, \mu^i(t) < \pi_A\}$. If t_i^{AA} is finite, then for each time $t \in (t_i^A, t_i^{AA}]$ firm i expects firm $-i$ to invest at some point in the future with positive probability, meaning $V^i(t) > 0$. And for each time $t > t_i^{AA}$ we have $\mu^i(t) < \pi_A$. Thus for all times $t > t_i^A$ we must have $f_i(t) > 0$ and $\lambda^i(t) = 0$. In this case firm $-i$ is in autarky with beliefs strictly above its autarky threshold beginning at time t_i^A, meaning $\lambda^{-i}(t) = \overline{\lambda}$ going forward. But then eventually firm i's posterior beliefs must drop below μ^*, at which point it cannot be optimal for firm i to invest immediately after obtaining a signal, a contradiction of $T_i^* = \infty$. So it must be that $t_i^{AA} = \infty$, i.e. $\lambda^{-i}(t) = 0$ for all $t > t_i^A$.

In that case $V^i(t_i^A) = 0$ and thus $f_i(t_i^A) = 0$. Now, if $f_i(t) \geq 0$ on some positive-measure subset of $[0, t_i^A]$, then for some $t' < t_i^A$ we must have $f_i(t') \geq 0$ and $f_i(t') > 0$, meaning that $f_i(t) > 0$ for $t > t'$ sufficiently small. But for f_i to decline back to zero by time t_i^A, there must be a positive-measure set of times at which f_i is both strictly positive and has a strictly negative derivative, a contradiction. So it must be that $f_i(t) < 0$ a.e. on $[0, t_i^A]$, i.e. $\lambda^i(t) = \overline{\lambda}$ for all such times. But then firm i prospects at the maximum rate at all times prior to t_i^A, meaning that $\mu^{-i}(t) \leq \mu^i(t)$ for such times, a contradiction of $t_i^A < t_{-i}^A$. We conclude that $t_1^A = t_2^A$. Let t^A be this common time.

Suppose first that $t^A = \infty$. Then each firm i must prospect at less than full intensity on a positive-measure set of times, meaning there exists a time t_i' at which $f_i(t_i') \geq 0$ and $f_i'(t) > 0$. Thus $f_i(t) > 0$ for $t > t_i'$ sufficiently small, and by reasoning similar to the previous paragraph $f_i(t) > 0$ for all $t > t_i'$. Thus $\lambda^i(t) = 0$ for $t > t_i'$, meaning firm $-i$ is in autarky with beliefs strictly above the autarky threshold. It therefore sets $\lambda^{-i}(t) = \overline{\lambda}$ for $t > t_i'$, contradicting $t_i^A = t^A = \infty$. So it must be that $t^A < \infty$.

Next, suppose that $f_i(t^A) > 0$ for some i. Then also $f_i(t) > 0$ for t sufficiently close to t^A, meaning $\lambda^i(t) = 0$ for such times. But then $\mu^{-i}(t)$ is constant on this interval, contradicting $t_{-i}^A = t^A$. So $f_i(t^A) = 0$ for each firm i. By now-familiar arguments, it must therefore be that $f_i(t) < 0$ for almost all $t < t^A$, i.e. $\lambda^i(t) = \overline{\lambda}$ for each i and a.e. $t < t^A$. Therefore $t^A = T^A$. Further, $f_i(t^A) = 0$ implies $V^i(t^A) = 0$, so $\lambda^{-i}(t) = 0$ for all $t > t^A$. Thus each firm must use the threshold prospecting strategy $\overline{T}_i = T^A$. □

We complete the proof by ruling out mixed prospecting rules in equilibrium. This is accomplished by the following lemma, which establishes that any equilibrium involving randomization over prospecting implies existence of a pure-strategy equilibrium involving interior prospecting. As no pure-strategy equilibria exhibit such behavior, no mixed-strategy equilibria exist.

Lemma D.15. *Fix any equilibrium in threshold investment strategies. Then there exists a payoff-equivalent equilibrium in pure strategies, exhibiting interior prospecting* *whenever some firm randomized over prospecting rates in the original equilibrium.*

Proof. Fix an equilibrium involved randomized prospecting, and fix a firm i. After time T_i^*, firm i's prospecting rule does not affect firm $-i$'s payoffs or incentives; thus λ^i may be replaced with any pure strategy maximizing i's payoffs subsequent to time T_i^* without disturbing the equilibrium. So consider times $t < T_i^*$.

Let

$$\Omega^i(t) \equiv \mathbb{E}\left[\exp\left(-\int_0^t \lambda^i(s)\,ds\right)\right]$$

be the ex ante probability that firm i has obtained no signal by time t. Define a new pure-strategy prospecting rule $\widetilde{\lambda}^i$ by letting $\widetilde{\lambda}^i(t) = -\frac{d}{dt}\log\Omega^i(t)$ for all times (with the prospecting rule arbitrary at any point of non-differentiability of Ω^i). By construction, λ^i and $\widetilde{\lambda}^i$ induce the same sequence of induce the same distribution of investment times by firm i, and thus the same posterior beliefs for firm $-i$ conditional on observing no investment. Therefore firm $-i$'s incentives are unchanged by replacing λ^i with $\widetilde{\lambda}^i$.

It remains to check that $\widetilde{\lambda}^i$ is feasible and optimal for firm i. Note that

$$\widetilde{\lambda}^i(t) = \frac{1}{\Omega^i(t)}\mathbb{E}\left[\lambda^i(t)\exp\left(-\int_0^t \lambda^i(s)\,ds\right)\right].$$

The second factor on the rhs is bounded above by $\overline{\lambda}\Omega^i(t)$ and below by zero, hence $\widetilde{\lambda}^i(t) \in [0,\overline{\lambda}]$, ensuring feasibility. As for optimality, suppose first that at time t, the action $\lambda^i(t)$ is strictly optimal for firm i. Then it must be non-random, in which case the previous expression for $\widetilde{\lambda}^i(t)$ collapses to $\widetilde{\lambda}^i(t) = \lambda^i(t)$. So at any times for which randomization is not optimal for firm i, the modified prospecting rule specifies the same prospecting intensity as the original rule. And at all other times, any prospecting intensity is optimal, thus in particular the intensity specified by $\widetilde{\lambda}^i$ is optimal. So $\widetilde{\lambda}^i$ is an optimal prospecting rule.

This argument shows that firm i's randomized prospecting rule may be replaced by a non-random one which is also optimal for firm i, without disturbing firm $-i$'s payoffs or incentives. This procedure may be performed for both firms, yielding a pure strategy equilibrium.

Finally, for any time t at which $\lambda^i(t)$ is not deterministic, it must be that $\Pr(\lambda^i(t) > 0) > 0$ and $\Pr(\lambda^i(t) < \overline{\lambda}) > 0$, in which case the previous expression for $\widetilde{\lambda}^i$ implies $\widetilde{\lambda}^i(t) \in (0,\overline{\lambda})$. So randomization in the original equilibrium implies an interior prospecting rate in the new equilibrium. □

D.6 Proof of Proposition 4

To prove the small-r result, we show that whenever $r \leq r^*$, welfare under the leader-follower equilibrium exceeds welfare in the symmetric equilibrium. Let $V^A(t) \equiv \frac{\overline{\lambda}}{\overline{\lambda}+r} K(\mu^{\overline{\lambda}}(t) - \pi_A)$ be the autarky payoff under beliefs $\mu^{\overline{\lambda}}(t)$. We first show that in the symmetric equilibrium, $V^i(t) = V^A(t)$ for each i and all $t \leq T^A$. First note that for all such times, $\mu^i(t) = \mu^{\overline{\lambda}}(t)$. Then trivially $V^i(T^A) = V^A(T^A)$, as at time T^A each firm is in autarky with beliefs $\mu^{\overline{\lambda}}(T^A)$. So evaluate the functional F^i defined in Appendix C at V^A for any time $t \leq T^A$. Using the identity $\overline{V} = \frac{\overline{\lambda}}{\overline{\lambda}+r} K(\pi_+ - \pi_A)$, which holds whenever $r \leq r^*$, as well as the identity $\widetilde{V}^i(t) - c = K(\mu^{\overline{\lambda}}(t) - \pi_A)$, which holds given that $T_i^* = \infty$, yields $F^i(V^A, t) = 0$. Then a standard verification argument establishes that $V^A(0) = V^S$.

Now consider the leader-follower equilibrium. Recall that when $r \leq r^*$, Lemma D.1 implies that $T_F^* = 0$. Hence the leader is in autarky for all times and $V^L = V^A(0)$. So consider the follower's strategy. Suppose firm i is the follower. Note that $\mu^i(t) = \mu^{\overline{\lambda}}(t)$ for all time, and that $\widetilde{V}^i(t) - c > K(\mu^{\overline{\lambda}}(t) - \pi_A)$ for all time given that investing immediately is strictly dominated by waiting at all times. Hence $F^i(V^A, t) < 0$ for all time. Then as V^A is a bounded function, a standard verification argument establishes that V^A is bounded strictly above by the payoff of the threshold strategy $\overline{T}_i = T_i^* = \infty$. Since V^F is an upper bound on the payoff of any strategy followed by firm i, it must be that $V^F > V^A(0)$. Thus $V^L + V^F > 2V^A(0) = 2V^S$, as claimed.

We now prove the large-r result. Going forward, we will assume that $r > r^*$. We first establish that $V^F > V^S > V^L$. Write $V^F(t)$, $V^S(t)$, $V^L(t)$ for the time-t continuation value of each firm in each equilibrium given no signal and no investment by the other firm. Let $\widehat{T}_F \equiv \min\{\overline{T}_F, T_F^*\}$ be the time at which the follower becomes passive in the leader-follower equilibrium. The leader's beliefs equal $\mu^{\overline{\lambda}}(\widehat{T}_F)$ at time $\widehat{T}_F$, and further the leader is in autarky going forward. It follows that $V^L(\widehat{T}_F) = V^A(\widehat{T}_F)$.

Meanwhile, a firm in the symmetric equilibrium possesses posterior beliefs $\mu^{\overline{\lambda}}(\widehat{T}_F)$ at time $\widehat{T}_F$ given that $\widehat{T}_F < T^A$ (as established in Proposition 2). Further, the autarky strategy is feasible but not optimal for that firm in the continuation after time $\widehat{T}_F$. This is because when $r > r^*$, investing following observation of investment by the other firm improves on the autarky strategy of ignoring the other firm's actions and continuing to prospect. And since $\widehat{T}_F < T^A$, each firm invests with positive probability subsequent to time $\widehat{T}_F$ in the symmetric equilibrium. It must therefore be that $V^S(\widehat{T}_F) > V^A(\widehat{T}_F) > V^L(\widehat{T}_F)$.

Next, observe that the follower could achieve the symmetric equilibrium continuation value at time $\widehat{T}_F$ by following the strategy of prospecting until time T^A, investing immediately if it has obtained a signal or observed investment, and then halting all prospecting and investment subsequent to time T^A, regardless of what it sees the other firm do. However, this strategy cannot be optimal, since the leader invests with positive probability after time T^A, and the follower's payoff would be improved by investing in such histories whenever it has not yet obtained a signal. It must therefore be that $V^F(\widehat{T}_F) > V^S(\widehat{T}_F)$.

To complete the argument, we show that $V^F(\widehat{T}_F) > V^S(\widehat{T}_F) > V^L(\widehat{T}_F)$ implies that $V^F > V^S > V^L$. Note that no firm in either equilibrium delays investment prior to time $\widehat{T}_F$, and posterior beliefs for all firms equal $\mu^{\overline{\lambda}}(t)$ for all times prior to $\widehat{T}_F$. It follows that $F^i(\cdot, t)$ (as defined in Appendix C) is the same for a leader, follower, or firm in the symmetric equilibrium prior to $\widehat{T}_F$. Then since $V^F(\widehat{T}_F) > V^S(\widehat{T}_F) > V^L(\widehat{T}_F)$, a standard comparison result implies that $V^F > V^S > V^L$.

To complete the proof, we perform a limiting payoff comparison as $r \to \infty$. For the remainder of the proof, we will make the dependence of variables on r explicit. Note in particular that $\overline{\mu}(r)$ and $\mu^*(r)$ are both functions of r, while π_A is independent of r. We begin with two auxiliary lemmas.

Lemma D.16. *For sufficiently large* r, $\max\{\mu^*(r), \pi_A\} < \overline{\mu}(r) < \pi_0$.

Further, $\lim_{r\to\infty} \overline{\mu}(r) = \pi_A$.

Proof. Note that as $r \to \infty$, $\Delta_I(\mu, r)$ converges uniformly to $-(\mu_+ R - 1)$ for all $\mu \in [\pi_-, \pi_0]$, and thus $\mu^*(r)$ approaches $\underline{\mu}$, where $\underline{\mu}$ solves $\underline{\mu}_+ R - 1 = 0$. Since $\pi_{A+} R - 1 > 0$ and $\pi_+ R - 1 > 0$, it must therefore be that $\mu^*(r) < \min\{\pi_0, \pi_A\}$ for large r. In particular, $\mu^*(r) < \pi_0$ implies that

$$\Delta_P(\mu^*(r), r) = \frac{\mu^*(r) - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r}(\pi_+ R - 1) - K(\mu^*(r) - \pi_A).$$

As the first term approaches zero for large r while $\mu^*(r) < \pi_A$ for large r, we must have $\Delta_P(\mu^*(r), r) > 0$, i.e. $\overline{\mu}(r) > \mu^*(r)$. Lemma D.6 then further implies that

$\overline{\mu}(r) > \pi_A$ for such r. Further, for large r and $\mu > \mu^*(r)$, $\Delta_P(\mu, r)$ converges uniformly to $-K(\mu - \pi_A)$, and thus $\overline{\mu}(r)$ converges to π_A. Since $\pi_A < \pi_0$, we therefore have $\overline{\mu} < \pi_0$ for r sufficiently large. □

Lemma D.17. $\lim_{r\to\infty} r(T^A - \overline{T}_F(r)) = (\pi_+ R - 1)/(h(\pi_+)(\pi_{++}R - 1) - c)$.

Proof. Recall that $T^A = (\mu^{\overline{\lambda}})^{-1}(\pi_A)$ while $\overline{T}_F(r) = (\mu^{\overline{\lambda}})^{-1}(\overline{\mu}(r))$. So to first order,

$$T^A - \overline{T}_F(r) = -\frac{1}{\dot{\mu}^{\overline{\lambda}}(T^A)}(\overline{\mu}(r) - \pi_A) + O((\overline{\mu}(r) - \pi_A)^2).$$

For large r, $\overline{\mu}(r) \in (\mu^*, \pi_0)$ and so $\overline{\mu}(r)$ solves

$$\frac{\mu - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda} + r}(\pi_+ R - 1) = K(\mu - \pi_A).$$

The solution to this equation may be written to first order in r^{-1} as

$$\overline{\mu}(r) = \pi_A + K^{-1}\frac{\overline{\lambda}(\pi_A - \pi_-)}{\pi_+ - \pi_-}(\pi_+ R - 1)r^{-1} + O(r^{-2}).$$

Thus

$$T^A - \overline{T}_F(r) = -K^{-1}\frac{\overline{\lambda}(\pi_A - \pi_-)}{\dot{\mu}^{\overline{\lambda}}(T^A)(\pi_+ - \pi_-)}(\pi_+ R - 1)r^{-1} + O(r^{-2}).$$

Now, using Lemma B.2 to eliminate $\dot{\mu}^{\overline{\lambda}}(T^A)$ yields

$$T^A - \overline{T}_F(r) = \frac{\pi_+ R - 1}{K(\pi_+ - \pi_A)}r^{-1} + O(r^{-2}) = \frac{\pi_+ R - 1}{h(\pi_+)(\pi_{++}R - 1) - c}r^{-1} + O(r^{-2}).$$

Multiplying through by r and taking $r \to \infty$ yields the desired identity. □

In light of Lemma D.16, going forward we will assume that r is sufficiently large that $\overline{T}_F(r) < T^*_F(r), T^A$. Fix a strategy profile in which both firms play the leader's strategy. Let $\nu^{\overline{\lambda}}(t)$ be the associated time-t probability that a firm's opponent has obtained no signal, supposing it hasn't invested yet. Further let $\pi_P(t) \equiv \overline{\lambda}K(\mu^{\overline{\lambda}}(t) - \pi_A)$ and $\pi_O(t) \equiv \nu^{\overline{\lambda}}(t)h(\pi_0)(\pi_+ R - 1)$ be each firm's time-t flow profits from prospecting and observing investment, respectively, conditional on having obtained no signal and having observed no investment. (Note that when $r > r^*$, each firm optimally invests immediately following observation of investment.) Finally, let $\delta(t) \equiv$

$(1-(1-e^{-\overline{\lambda}t})h(\pi_0)$ be a firm's probability of reaching time t without having observed investment.

Each firm's profits in each equilibrium may be written using this notation. Symmetric equilibrium profits are

$$V^S(r) = \int_0^{T^A} dt\, e^{-(r+\overline{\lambda})t}\delta(t)(\pi_P(t) + \pi_O(t)),$$

where the upper limit of integration accounts for the termination of flow profits at time T^A supposing no firm has acquired a signal or invested by that time. Meanwhile the leader's profits are

$$V^L(r) = \int_0^{\overline{T}_F(r)} dt\, e^{-(r+\overline{\lambda})t}\delta(t)\left(\pi_P(t) + \pi_O(t)\right) + e^{-(r+\overline{\lambda})\overline{T}_F(r)}\delta(\overline{T}_F(r))\frac{\pi_P(\overline{T}_F(r))}{\overline{\lambda}+r},$$

where the final term accounts for the transition to autarky supposing no firm has acquired a signal or invested by time $\overline{T}_F(r)$. (Recall that $\overline{T}_F(r) < T_F^*(r)$, so $\overline{T}_F(r)$ is the time of transition to autarky.) Finally, the follower's profits are

$$V^L(r) = \int_0^{\overline{T}_F(r)} dt\, e^{-(r+\overline{\lambda})t}\delta(t)\left(\pi_P(t) + \pi_O(t)\right) + \int_{\overline{T}_F(r)}^{\infty} dt\, e^{-rt}e^{-\overline{\lambda}\overline{T}_F(r)}\delta(t)\pi_O(t),$$

where the final term accounts for the termination of prospecting at time $\overline{T}_F(r)$.

Define $\Delta V(r) \equiv re^{r\overline{T}_R(r)}(2V^S(r) - V^L(r) - V^F(r))$. This expression may be written explicitly as

$$\begin{aligned}\Delta V(r) = 2&\int_{\overline{T}_F(r)}^{T^A} dt\, re^{-r(t-\overline{T}_F(r))}e^{-\overline{\lambda}t}\delta(t)(\pi_P(t) + \pi_O(t)) \\ &- \int_{\overline{T}_F(r)}^{\infty} dt\, re^{-r(t-\overline{T}_F(r))}e^{-\overline{\lambda}\overline{T}_F(r)}\delta(t)\pi_O(t) \\ &- \frac{r}{\overline{\lambda}+r}e^{-\overline{\lambda}\overline{T}_F(r)}\delta(\overline{T}_F(r))\pi_P(\overline{T}_F(r)).\end{aligned}$$

We now take the limit $r \to \infty$. Recall that $\lim_{r\to\infty}\overline{T}_F(r) = T_A$ and $\pi_P(T^A) = 0$. Thus the final term vanishes in the limit. To evaluate the integrals, make the substitution $t' = r(t - \overline{T}_F(r))$. As π_P, π_O, and δ are bounded functions, the resulting integrands are uniformly bounded for all t' and r, and the bounded convergence theorem may be used to evaluate each integral in the limit. The first converges to

$2(1-\exp(-\lim_{r\to\infty} r(T^A-\overline{T}_F(r))))e^{-\bar{\lambda}T^A}\delta(T^A)\pi_O(T^A)$ while the second converges to $-e^{-\bar{\lambda}T^A}\delta(T^A)\pi_O(T^A)$. Combining these calculations and invoking Lemma D.17 yields

$$\lim_{r\to\infty}\Delta V(r) = \left(1-2\exp\left(-\frac{\pi_+R-1}{h(\pi_+)(\pi_{++}R-1)-c}\right)\right)e^{-\bar{\lambda}T^A}\delta(T^A)\pi_O(T^A).$$

The sign of $2V^S(r)-V^L(r)-V^F(r)$ for large r must be the same as the sign of $\lim_{r\to\infty}\Delta V(r)$. Hence it is strictly positive whenever $c > \underline{c} \equiv h(\pi_+)(\pi_{++}R-1)-(\pi_+R-1)/\log 2$. Note that $\underline{c}$ is independent of r and $\underline{c} < \bar{c}$ given that $\log 2 < 1$, as claimed in the proposition statement.